

Math 241 (Camp)

Centriod, Moments, Center of Mass and Moments of Inertia of Lamina

- **Area and the Centriod of a Lamina** A *lamina* is an arbitrarily thin, flat solid. Since it's thin, we can approximate it by a 2D domain.

The area of a lamina occupying a domain D in \mathbb{R}^2 is

$$A = \iint_D dA.$$

The centriod, or center of area, is given by (\tilde{x}, \tilde{y}) where

$$\tilde{x} = \frac{1}{A} \iint_D x \, dA = \frac{\iint_D x \, dA}{\iint_D dA} \quad \text{is the average value of } x \text{ in } D,$$

and

$$\tilde{y} = \frac{1}{A} \iint_D y \, dA = \frac{\iint_D y \, dA}{\iint_D dA}, \quad \text{is the average value of } y \text{ in } D.$$

- **Mass, Moments and Center of Mass of a Lamina**

Now let's assume that the lamina has a density function denoted $\rho(x, y)$. Then the *total mass* of the lamina is

$$m = \iint_D \rho(x, y) \, dA.$$

The *center of mass* is the point at which the mass is effectively concentrated for purposes of the lamina's response to external forces. (It is also the point at which you can balance the lamina.) We determine its location by finding the first *moments*:

$$M_x = \iint_D y \rho(x, y) \, dA \quad \text{1st moment about the } x\text{-axis}$$

and

$$M_y = \iint_D x \rho(x, y) \, dA \quad \text{1st moment about the } y\text{-axis}$$

The center of mass is then determined by dividing these moments by the mass:

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right).$$

Note that

$$(\bar{x}, \bar{y}) = \left(\frac{\iint_D x \rho(x, y) \, dA}{\iint_D \rho(x, y) \, dA}, \frac{\iint_D y \rho(x, y) \, dA}{\iint_D \rho(x, y) \, dA} \right);$$

i.e., \bar{x} and \bar{y} are the density-weighted averages of x and y , respectively, in D .

If the lamina has constant density, then the centriod and the center-of-mass are located at the same point.

- **Moments of Inertia of a Lamina** The *moments of inertia* (or second moments) determine the resistance to rotational accelerations about the given axis or point:

– moment of inertia about the x -axis:

$$I_x = \iint_D y^2 \rho(x, y) \, dA$$

– moment of inertia about the y -axis:

$$I_y = \iint_D x^2 \rho(x, y) \, dA$$

– moment of inertia about the origin (*i.e.*, the z -axis):

$$I_0 = \iint_D (x^2 + y^2) \rho(x, y) \, dA = I_x + I_y$$

The *radius of gyration of a lamina about an axis* is the determined by

$$R^2 = \frac{I}{m}$$

where I is the moment of inertia about that axis and R is the associated radius of gyration. If the mass of the lamina were concentrated at a distance R from the given axis, then the moment of inertia with respect to that axis would be unchanged. So

$$\bar{\bar{x}}^2 = \frac{I_y}{m} = \frac{1}{m} \iint_D x^2 \rho(x, y) \, dA,$$

$$\bar{\bar{y}}^2 = \frac{I_x}{m} = \frac{1}{m} \iint_D y^2 \rho(x, y) \, dA,$$

$$\bar{\bar{r}}^2 = \frac{I_0}{m} = \bar{\bar{x}}^2 + \bar{\bar{y}}^2.$$

I.e., $\bar{\bar{x}}$ and $\bar{\bar{y}}$ are the positive square roots of the density-weighted averages of x^2 and y^2 , respectively, on D .

Given a lamina of density $\rho(x, y) = y$ on the region bounded by $y = e^x$, $y = 0$, $x = 0$ and $x = 1$, do the following:

1. Sketch the region occupied by the lamina. Label all boundary curves with their algebraic definitions. Also label the $y = e^x$ curve with its alternate definition in the form $x = g(y)$.

2. Algebraically define the region occupied by the lamina using both orders, *i.e.*, both as a x -simple region and as a y -simple region. *Note: for one of these orders, the region must be defined piecewise.*

3. Using either definition of the domain D , find the mass of the lamina.

More on next page.

4. Find the first moments and the center of mass of the lamina.

5. Find the moments of inertia and the radii of gyration of the lamina about the x -axis, y -axis and the origin.