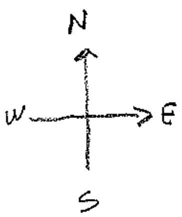
Directional Derivatives and the Gradient

You are standing on a hill whose elevations are given by $z = f(x, y) = x^2y^3$ where x, y, z are in meters; it's a strange-shaped hill. You are standing at the point $(-1, 2, 8)$ on the hill. Hint: you might want to find the gradient at your location before answering the following questions.

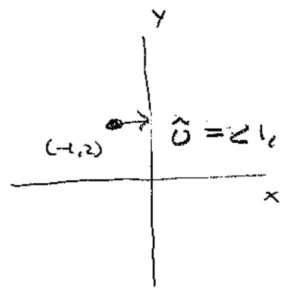
$$\nabla f = \langle f_x, f_y \rangle = \langle 2xy^3, 3x^2y^2 \rangle$$

note: "location" is the domain is $(x,y) = (-1,2)$; $(-1,2,8)$ is the point $(x, y, f(x,y))$ on the surface.

$$\nabla f(-1,2) = \langle -16, 12 \rangle$$



1. What is the slope you are facing if you are looking due east? Are you facing uphill or downhill?



Two (related) ways to do this

i) East is positive x direction

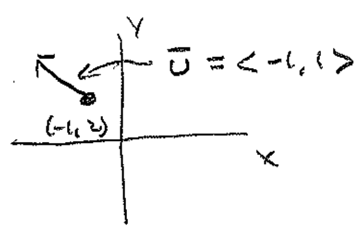
$\Rightarrow$ slope is $\left.\dfrac{\partial f}{\partial x}\right|_{(-1,2)} = 2xy^3\Big|_{(-1,2)} = -16$

OR ii) East is the direction $\hat{u} = \langle 1, 0 \rangle$

Directional Derivative is $D_u f = \nabla f\big|_{(-1,2)} \cdot \hat{u} = \langle -16, 12 \rangle \cdot \langle 1, 0 \rangle = -16$

Either way, $16 < 0 \Rightarrow$ facing downhill.

2. If you are facing northwest, are you facing uphill or downhill? What is the slope in that direction?



NW direction $\Rightarrow \vec{u} = \langle -1, 1 \rangle$

Need unit vector $\hat{u} = \dfrac{\vec{u}}{\|\vec{u}\|} = \dfrac{\langle -1, 1 \rangle}{\sqrt{(-1)^2 + 1^2}} = \left\langle -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$

$$D_u f = \nabla f\big|_{(-1,2)} \cdot \hat{u} = \langle -16, 12 \rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{16}{\sqrt{2}} + \frac{12}{\sqrt{2}} = \frac{28}{\sqrt{2}} = 14\sqrt{2} > 0$$

same location, same gradient

new direction

slope is $14\sqrt{2}$ uphill

3. In what direction should you face to be looking at the steepest uphill? What is the slope in that direction?

$\bar{u} = \nabla f = \langle -16, 12 \rangle$, slope is $\|\nabla f\| = \sqrt{(-16)^2 + 12^2} = 4\sqrt{4^2+3^2} = 4 \cdot 5 = 20$

Note: slope ≥ 0 as expected

The direction can be rescaled (normalized) by multiplying by any positive number

e.g. $\bar{u} = \langle -4, 3 \rangle$ or $\hat{u} = \langle -\frac{4}{5}, \frac{3}{5} \rangle$ are valid answers for the direction. But slope requires the actual gradient (unscaled)

$\|\nabla f\| = \|\langle -16, 12 \rangle\|$

4. In what direction(s) should you start to head if you do not want to change your elevation?

We want to stay on a level curve $\Rightarrow$ initially head perpendicular ($\perp$) to the gradient.

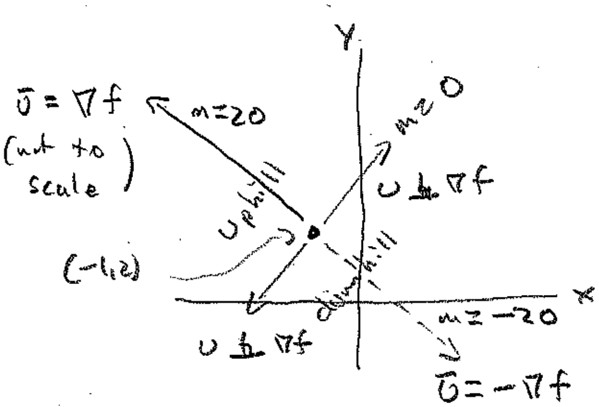
Note: $\langle a,b \rangle \perp \langle -b, a \rangle$ and $\langle a,b \rangle \perp \langle b, -a \rangle$

since $\langle a,b \rangle \cdot \langle -b, a \rangle = -ab + ba = 0$

So we want a direction $\bar{u}$ such that $\nabla f \cdot \bar{u} = 0$

$\nabla f = \langle -16, 12 \rangle$, so $\bar{u} = \langle 12, 16 \rangle$ or $\bar{u} = \langle -12, -16 \rangle$

Check: $D_u f = \nabla f \cdot \hat{u} = \langle -16, 12 \rangle \cdot \frac{\langle 12, 16 \rangle}{20} = 0$ ✓



$m \equiv$ slope