**Math 241 (Camp)**

## Conservative Vector Fields

A vector field, $\mathbf{F}$, is a gradient vector field if $\mathbf{F} = \nabla f$ for some scalar field, $f$; $f$ is often called a potential function of $\mathbf{F}$. Gradient vector fields are very important in both mathematical and physical problems. For example, the fundamental theorem of calculus can be extend to line integrals of gradient vector fields.

**Theorem 1: The Fundamental Theorem of Calculus for Line Integrals:** Suppose that $f$ is a differentiable scalar function and $C$ is a piecewise smooth curve, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

where $\mathbf{F} = \nabla f$ and $C$ is parameterized by $\mathbf{r}(t)$, $a \leq t \leq b$.

We need to be able to determine when a vector field can be written as a gradient. We will first discuss this for 2D vector fields, $\mathbf{F}(x, y)$, then extend this to 3D vector fields, $\mathbf{F}(x, y, z)$.

**Theorem 2: Conservative Fields in $\mathbb{R}^2$:** Let $D$ be an open simply-connect region in $\mathbb{R}^2$ and let $\mathbf{F} = P(x, y)\,\hat{\imath} + Q(x, y)\,\hat{\jmath}$ be a differentiable field defined on $D$. The following conditions on $\mathbf{F}$ are all equivalent:

i. For any oriented simple closed curve $C$ in $D$, $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

ii. For any two oriented simple curves, $C_1$ and $C_2$, in $D$ that have the same endpoints,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r};$$

when this holds, the line integral is called **path independent**.

iii. $\mathbf{F}$ is the gradient of some function $f$ in $D$; *i.e.,* $\mathbf{F} = \nabla f$ in $D$.

iv. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ in $D$.

A vector field satisfying one (and, therefore, all) of the conditions (i)-(iv) is called a **conservative vector field**.

**Proof** We shall prove Theorem 2 cyclically, *i.e.,* we will separately prove 4 implications: (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i). If we then assume any one of the four statements is true, we can conclude that any of the remaining statements must then be true; hence, the four statements are all equivalent.

1. (i) $\Rightarrow$ (ii): Let $C_1$ and $C_2$ be any two oriented simple curves in $D$ with the same endpoints. Let $-C_2$ denote the curve $C_2$ with a reversed orientation. Then $C = C_1 \cup (-C_2)$ is a simple closed curve in $D$. By hypothesis (1), $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$; therefore,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

$$\Longrightarrow \quad \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

2. (ii) $\Rightarrow$ (iii): Assume $C$ is an oriented simple curve in $D$ connecting a fixed point $(a, b)$ in $D$ to any arbitrary point $(x, y)$ in $D$. Let $f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r}$. By hypothesis (2), $f(x, y)$ is independent of $C$. Therefore, we can define a new path $C_1$, consisting of the two line segments connecting $(a, b)$ to $(x, b)$ and $(x, b)$ to $(x, y)$,[1] such that

$$f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_a^x P(t, b)\ dt + \int_b^y Q(x, t)\ dt$$

$$\Longrightarrow \quad \frac{\partial f}{\partial y}(x, y) = Q(x, y)\,.$$

Similarly, we can define another path, $C_2$, connecting $(a, b)$ to $(x, y)$ consisting of the two line segments connecting $(a, b)$ to $(a, y)$ and $(a, y)$ to $(x, y)$. Then, invoking Clairaut's Theorem, we get

$$f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_b^y Q(a, t)\ dt + \int_a^x P(t, y)\ dt$$

$$\Longrightarrow \quad \frac{\partial f}{\partial x}(x, y) = P(x, y)\,.$$

Therefore $\mathbf{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} = \dfrac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \dfrac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} = \nabla f$.

3. (iii) $\Rightarrow$ (iv): Assume $\mathbf{F} = \nabla f$ in $D$ for some differentiable scalar field, $f(x, y)$, where $\mathbf{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}$. Then

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial Q}{\partial x}.$$

4. (iv) $\Rightarrow$ (i): Let $C$ be any simple closed curve in $D$ and define $D_1$ to be the simply connected region contained by $C$, *i.e.*, $C = \partial D_1$. Since $D$ is simply connected,

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[1] You might find useful to sketch $C_1$.

$D_1 \subset D$; therefore, by hypothesis (4), $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial y}$ for all points in $D_1$.[2] Invoking Green's Theorem, we get

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = 0.$$

The theorem defining conservative fields in $\mathbb{R}^3$ is quite similar. The two differences are the use of the curl of $F$ in the fourth statement and the less restrictive requirements on the domain in the hypothesis.

**Theorem 3: Conservative Fields in $\mathbb{R}^3$**: Let $\mathbf{F} = P(x,y,z)\,\hat{\boldsymbol{\imath}} + Q(x,y,z)\,\hat{\boldsymbol{\jmath}} + R(x,y,z)\,\hat{\boldsymbol{k}}$ be a differentiable field defined on $\mathbb{R}^3$, except possibly for a finite number of points. The following conditions on $\mathbf{F}$ are all equivalent:

i. For any oriented simple closed curve $C$, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

ii. **Path Independence**:
For any two oriented simple curves, $C_1$ and $C_2$, that have the same endpoints,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

iii. **Existence of a potential function**:
$\mathbf{F}$ is the gradient of some function $f$; *i.e.*, $\mathbf{F} = \nabla f$.

iv. $\nabla \times \mathbf{F} = \mathbf{0}$.

A vector field satisfying one (and, therefore, all) of the conditions (i)-(iv) is called a **conservative vector field**.

Theorem 3 can be proven in a similar way to the proof given for Theorem 2. Key differences:

- (ii) ⇒ (iii): use 3 different curves, each of which consists of 3 line segments parallel to the coordinate axes.
- (iii) ⇒ (iv): use the vector identity, $\nabla \times \nabla f = 0$.
- (iv) ⇒ (iv): invoke Stokes' Theorem (instead of Green's Theorem) for an oriented surface $S$ whose boundary is $\partial S = C$ .

See a vector calculus text for more details; *e.g.*,
*Vector Calculus*, Marsden and Tromba, 5 ed., W. H. Freeman and Co., 2003.

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[2] This statement creates the requirement that $D$ be a simply connected region in $\mathbb{R}^2$; otherwise, $D_1$ might not be a subset of $D$ and the equality might not hold for all points in $D_1$.