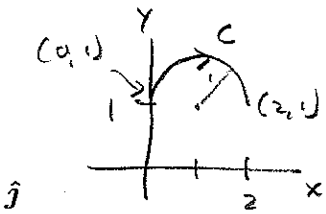
Math 241 (Camp)

Conservative Vector Field: Worksheet

Consider the line integral

$$I = \int_C \mathbf{F} \cdot \mathbf{dr}$$



where

$$\mathbf{F}(x, y) = P(x, y)\,\hat{\imath} + Q(x, y)\,\hat{\jmath} = y^2\,\hat{\imath} + \left(2xy + 4y^3\right)\hat{\jmath}$$

and C is the upper semi-circle, $(x-1)^2 + (y-1)^2 = 1$, $y \geq 1$, from $(0,1)$ to $(2,1)$. We could parameterize C and calculate the line integral directly, but instead we'll show that $\mathbf{F}$ is conservative and then find 2 different ways to evaluate the line integral.

1. Show that $\mathbf{F}(x, y)$ is conservative for all $(x, y) \in \mathbb{R}^2$.
   I.e., show that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ *for all* $(x, y) \in \mathbb{R}^2$ *and invoke the theorem defining conservative fields.*

   $$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y^2) = 2y$$

   $$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y = \frac{\partial P}{\partial y}$$ for all (x,y)

2. The first method uses the existence of a potential for conservative fields and then invokes the fundamental theorem to evaluate the line integral.

   (a) Find a potential function for $\mathbf{F}$; *i.e.*, find ϕ such that $\mathbf{F}(x, y) = \nabla\phi(x, y)$ or in component form $\langle P, Q\rangle = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right\rangle$. A 2 step procedure:

      i. Since $P(x, y) = \frac{\partial \phi}{\partial x}(x, y)$, we can let $\phi(x, y) = \int P(x, y)\,dx + g(y)$ where $g(y)$ is an arbitrary function of y only – currently unknown.

      ii. To find $g(y)$, since $Q(x, y) = \frac{\partial \phi}{\partial y}$, differentiate the above result and show that $g'(y) = Q(x, y) - \frac{\partial}{\partial y}\left[\int P(x, y)\,dx\right]$ and then integrate with respect to y. Then $\phi = \int P(x, y)\,dx + g(y) + C$ where C is an arbitrary constant.

$$\phi = \int P\,dx = \int y^2\,dx = xy^2 + g(y)$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = Q(x,y) \Rightarrow \underline{2xy} + g'(y) = \underline{2xy} + 4y^3 \Rightarrow g'(y) = 4y^3 \quad (y \text{ only!})$$

These cancel, if they don't either you've made an error or F was not conservative.

$$\Rightarrow g(y) = \int 4y^3\,dy = y^4 + C$$

$$\Rightarrow \phi(x,y) = xy^2 + y^4 + C$$

Potential function are unique only up to an additive constant; they are the analog of anti-derivatives for conservative fields.

(b) The fundamental theorem states that if $\mathbf{F}$ is conservative and $\mathbf{F} = \nabla\phi$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{C}} \nabla\phi \cdot \mathbf{dr} = \phi(\mathbf{b}) - \phi(\mathbf{a})$$

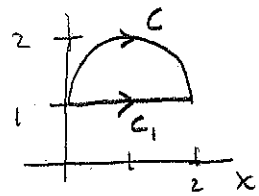
where C is a path from $\mathbf{r} = \mathbf{a}$ to $\mathbf{r} = \mathbf{b}$. Use the ϕ found about to evaluate the given line integral.

$\bar{b} = (2,1) \Rightarrow \phi(\bar{b}) = \phi(2,1) = 2\cdot 1^2 + 1^4 = 3$

$\bar{a} = (0,1) \Rightarrow \phi(\bar{a}) = \phi(0,1) = 0\cdot 1^2 + 1^4 = 1$

$\Rightarrow \int_C \bar{F}\cdot d\bar{r} = 3 - 1 = 2$

3. If $\mathbf{F}$ is conservative then, instead of using the potential for $\mathbf{F}$, we can use the existence of path independence for C and move to a path which is easier to parameterize. Here, we'll replace C by C_1 where C_1 is the horizontal line from $(0,1)$ to $(2,1)$; since F is conservative for all $(x,y) \in \mathbb{R}^2$,



$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}.$$

(a) Parameterize C_1

(b) Use this parameterization to calculate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ directly.

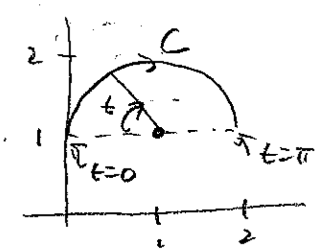
$C_1: \bar{r}(t) = \langle t, 1\rangle \quad 0 \le t \le 2$

$\Rightarrow \bar{r}'(t) = \langle 1, 0\rangle$

$\bar{F}(\bar{r}(t)) = \bar{F}(t,1) = \langle 1, 2t+4\rangle$

$\Rightarrow \int_C \bar{F}\cdot d\bar{r} = \int_0^2 \langle 1, 2t+4\rangle \cdot \langle 1, 0\rangle\, dt = \int_0^2 1\, dt = t\big|_0^2 = 2$ ✓

Extra: direct calculation $\quad C: r(t) = \langle 1-\cos t, 1+\sin t\rangle \quad 0 \le t \le \pi$



$\Rightarrow \bar{r}'(t) = \langle \sin t, -\cos t\rangle$

$\bar{F}(\bar{r}(t)) = \langle (1+\sin t)^2, 2(1-\cos t)(1+\sin t) + 4(1+\sin t)^3\rangle$

$\Rightarrow \int_C \bar{F}\cdot d\bar{r} = \int_0^{\pi} (1+\sin t)^2 \sin t - [2(1-\cos t)(1+\sin t) + 4(1+\sin t)^3]\cos t\, dt$

= Much more complicated!