Math 241 (Camp)

Conservative Vector Field: Worksheet

Consider the line integral

$$I = \int_C \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F}(x,y) = P(x,y)\,\hat{\boldsymbol{\imath}} + Q(x,y)\,\hat{\boldsymbol{\jmath}} = y^2\,\hat{\boldsymbol{\imath}} + \left(2xy + 4y^3\right)\hat{\boldsymbol{\jmath}}$$

and C is the upper semi-circle, $(x-1)^2 + (y-1)^2 = 1$, $y \geq 1$, from $(0,1)$ to $(2,1)$. We could parameterize C and calculate the line integral directly, but instead we'll show that $\mathbf{F}$ is conservative and then find 2 different ways to evaluate the line integral.

1. Show that $\mathbf{F}(x,y)$ is conservative for all $(x,y) \in \mathbb{R}^2$.
   I.e., show that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ *for all* $(x,y) \in \mathbb{R}^2$ *and invoke the theorem defining conservative fields.*

2. The first method uses the existence of a potential for conservative fields and then invokes the fundamental theorem to evaluate the line integral.

   (a) Find a potential function for $\mathbf{F}$; *i.e.*, find ϕ such that $\mathbf{F}(x,y) = \nabla\phi(x,y)$ or in component form $\langle P,\, Q\rangle = \left\langle \frac{\partial \phi}{\partial x},\, \frac{\partial \phi}{\partial y}\right\rangle$. A 2 step procedure:

      i. Since $P(x,y) = \frac{\partial \phi}{\partial x}(x,y)$, we can let $\phi(x,y) = \int P(x,y)\,dx + g(y)$ where $g(y)$ is an arbitrary function of y only – currently unknown.

      ii. To find $g(y)$, since $Q(x,y) = \frac{\partial \phi}{\partial y}$, differentiate the above result and show that $g'(y) = Q(x,y) - \frac{\partial}{\partial y}\left[\int P(x,y)\,dx\right]$ and then integrate with respect to y. Then $\phi = \int P(x,y)\,dx + g(y) + C$ where C is an arbitrary constant.

(b) The fundamental theorem states that if $\mathbf{F}$ is conservative and $\mathbf{F} = \nabla\phi$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{C}} \nabla\phi \cdot \, \mathbf{dr} = \phi(\mathbf{b}) - \phi(\mathbf{a})$$

where C is a path from $\mathbf{r} = \mathbf{a}$ to $\mathbf{r} = \mathbf{b}$. Use the ϕ found about to evaluate the given line integral.

3. If $\mathbf{F}$ is conservative then, instead of using the potential for $\mathbf{F}$, we can use the existence of path independence for C and move to a path which is easier to parameterize. Here, we'll replace C by C_1 where C_1 is the horizontal line from $(0,1)$ to $(2,1)$; since F is conservative for all $(x,y) \in \mathbb{R}^2$,

$$\int_C \mathbf{F} \cdot \, d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \, d\mathbf{r}.$$

(a) Parameterize C_1

(b) Use this parameterization to calculate $\int_{C_1} \mathbf{F} \cdot \, d\mathbf{r}$ directly.

4. EXTRA: Evaluate the line integral directly by parameterizing the semi-circle.