Problem 1 (10pts) Definitions and Axioms. Precision counts.

(a–5pts) State the Well Ordering Axiom.

Solution. If $S \subseteq \mathbb{N}$ is nonempty, then there is a minimal element in $S$, that is, there is $t \in S$ such that $t \leq t'$ for all $t' \in S$.

(b–5pts) Complete the following definition. Let $S$ be a non-empty set. Then $\mathscr{A} \subseteq \mathcal{P}(S)$ is a partition of $S$ if . . .

Solution. the following three conditions hold:

i) $\emptyset \notin \mathscr{A}$

ii) $\forall A, B \in \mathscr{A}$, $A = B$ or $A \cap B = \emptyset$, and

iii) $\bigcup_{A \in \mathscr{A}} A = S$.

Problem 2 (16pts) Computations: Consider the sets $S = \{1, 2, 3\}$ and $T = \{2, 3, 4, 5, 6\}$.

(a–4pts) Give a relation $R \subseteq S \times S$ which is a partial order on $S$ but not a total order.

Solution. There are many possibilities. For example, the set $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3)\}$ is a partial order and not a total order (it also fails to be an equivalence relations).

(b–4pts) Give a relation $R \subseteq S \times S$ which is a total order on $S$.

Solution. Again there are many possibilities. The set $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)\}$ is a total order (this corresponds with $\leq$ on $S$, by the way).

(c–4pts) Give a relation $R \subseteq S \times T$ which describes a function from $S$ to $T$.

Solution. One such example is $R = \{(1, 4), (2, 2), (3, 2)\}$.

(d–4pts) Give the equivalence relation $R \subseteq T \times T$ which corresponds with the partition $\mathscr{A} = \{\{2, 4, 6\}, \{3, 5\}\}$ of $T$.

Solution. $R = \{(2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 4), (4, 2), (2, 6), (6, 2), (4, 6), (6, 4), (3, 5), (5, 3)\}$. 
**Problem 3** (10pts) Prove that $\sum_{i=1}^{n}(2i - 1) = n^2$ for all $n \in \mathbb{N}$.

*Solution.* Let $S$ be the set

$$S = \left\{ n \in \mathbb{N} \mid \sum_{i=1}^{n}(2i - 1) = n^2 \right\}.$$ 

Obviously $1 \in S$ since

$$\sum_{i=1}^{1}(2i - 1) = (2(1) - 1) = 1 = 1^2.$$ 

Now suppose that $n \in S$, that is, that

$$\sum_{i=1}^{n}(2i - 1) = n^2.$$ 

Then

$$\sum_{i=1}^{n+1}(2i - 1) = \sum_{i=1}^{n}(2i - 1) + (2(n + 1) - 1) = n^2 + 2n + 1 = (n + 1)^2,$$

where the second to last equality is by the induction hypothesis. We conclude that $n + 1 \in S$. By the Principal of Mathematical Induction it follows that $S = \mathbb{N}$, so that $\sum_{i=1}^{n}(2i - 1) = n^2$ for all $n \in \mathbb{N}$ as required.

**Problem 4** (10pts) Let $F_i$ be the Fibonacci numbers defined as usual, so $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. Prove that $F_n = 2F_{n+2} - F_{n+3}$ for all $n \in \mathbb{N}$.

*Solution.* We’ll need the first few Fibonacci numbers, which are $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, and $F_5 = 5$.

So let $S$ be the set

$$S = \{n \in \mathbb{N} \mid F_n = 2F_{n+2} - F_{n+3}\}.$$ 

Obviously $1 \in S$ since $F_1 = 1 = 2(2) - 3 = 2F_3 - F_4$.

Now suppose that $\{1, \ldots, n\} \subseteq S$. If $n = 1$, then $n + 1 = 2 \in S$ since $F_2 = 1 = 2(3) - 5 = 2F_4 - F_5$.

If $n > 1$, then by the induction hypothesis (since $n, n - 1 \in S$), we have

$$F_n = 2F_{n+2} - F_{n+3}$$

and

$$F_{n-1} = 2F_{n+1} - F_{n+2}$$

whence

$$F_{n+1} = F_n + F_{n-1} = (2F_{n+2} - F_{n+3}) + (2F_{n+1} - F_{n+2})$$

$$= 2(F_{n+2} + F_{n+1}) - (F_{n+3} + F_{n+2}) = 2F_{n+3} - F_{n+4},$$

that is, $n + 1 \in S$. We conclude that $S = \mathbb{N}$ and thus that $F_n = 2F_{n+2} - F_{n+3}$ for all $n \in \mathbb{N}$ as required.

**Problem 5** (10pts) Suppose that $S$ is a non-empty set and $\sim$ is a relation on $S$ which is symmetric, transitive, and has the following interesting property: for each $a \in S$, there is some $b \in S$ such that $a \sim b$. Prove that $\sim$ is an equivalence relation.

*Solution.* Since $\sim$ is symmetric and transitive by hypothesis, we need only show that $\sim$ is reflexive, that is, that $a \sim a$ for all $a \in S$. Given $a \in S$, let $b \in S$ be such that $a \sim b$. Since $\sim$ is symmetric, $b \sim a$ as well. Then since $\sim$ is transitive, $a \sim b \land b \sim a$ implies $a \sim a$ as required.
Problem 6 (14pts) In class we discussed the relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ given by $(a, b) \sim (c, d)$ if $ad = bc$.

(a–10pts) Prove that $\sim$ is transitive.

Solution. Suppose that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ for some $(a, b), (c, d), (e, f) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Then $(a, b) \sim (c, d)$ implies $ad = bc$ while $(c, d) \sim (e, f)$ yields $cf = de$. So

$$(ad)f = (bc)f = b(cf) = b(de)$$

and thus $(af)d = (be)d$. Since $d \neq 0$, we can cancel it from both sides of this equation yielding $af = be$. We conclude that $(a, b) \sim (e, f)$ as required.

(b–4pts) Give 4 distinct elements of $[(1, 2)]_\sim$.

Solution. Here $(1, 2), (2, 4), (3, 6), (4, 8) \in [(1, 2)]_\sim$ (of course infinitely many other examples are possible).