Math 306, Linear Algebra II, Winter 2014
Homework 2, due Tuesday 1/14

Read sections:
1. 1.4 (for 1/10)

Do the following problems:
1. 1.3.8: Determine whether the following sets are subspaces of \( \mathbb{R}^3 \) under the operations of addition and scalar multiplication defined on \( \mathbb{R}^3 \). Justify your answers.
   (a) \( W_1 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2 \} \)
   (b) \( W_2 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2 \} \)
   (c) \( W_3 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0 \} \)
   (d) \( W_4 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0 \} \)
   (e) \( W_5 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1 \} \)
   (f) \( W_6 = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1^2 - 3a_2^2 = 6a_3^2 = 0 \} \)

2. 1.3.10: Prove that \( W_1 = \{ (a_1, a_2, \ldots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 0 \} \) is a subspace of \( F^n \), but \( W_2 = \{ (a_1, a_2, \ldots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 1 \} \) is not.

3. 1.3.11: Is the set \( W = \{ f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n \} \) a subspace of \( P(F) \) if \( n \geq 1 \)? Justify your answer.

4. 1.3.13: Let \( S \) be a nonempty set and \( F \) a field. Prove that for any \( s_0 \in S, \{ f \in F(S, F) : f(s_0) = 0 \} \) is a subspace of \( F(S, F) \).

5. 1.3.18: Prove that a subset \( W \) of a vector space \( V \) is a subspace of \( V \) if and only if \( 0 \in W \) and \( ax + y \in W \) whenever \( a \in F \) and \( x, y \in W \).

6. 1.3.24: Show that \( F^n \) is the direct sum of the subspaces \( W_1 = \{ (a_1, a_2, \ldots, a_n) \in F^n : a_n = 0 \} \) and \( W_2 = \{ (a_1, a_2, \ldots, a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0 \} \).

7. 1.3.25: Let \( W_1 \) denote the set of all polynomials \( f(x) \) in \( P(F) \) such that in the representation
   \[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \]
   we have \( a_i = 0 \) whenever \( i \) is even. Likewise let \( W_2 \) denote the set of all polynomials \( g(x) \) in \( P(F) \) such that in the representation
   \[ g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0, \]
   we have \( b_i = 0 \) whenever \( i \) is odd. Prove that \( P(F) = W_1 \oplus W_2 \).

8. 1.3.30: Let \( W_1 \) and \( W_2 \) be subspaces of a vector space \( V \). Prove that \( V \) is the direct sum of \( W_2 \) and \( W_2 \) if and only if each vector in \( V \) can be uniquely written as \( x_1 + x_2 \), where \( x_1 \in W_1 \) and \( x_2 \in W_2 \).

9. Suppose that \( V \) is a vector space over a field \( F \) with subspaces \( U \) and \( W \). In addition, suppose that \( U \oplus W \) is a subspace of \( V \). Prove that \( U \subseteq W \text{ or } W \subseteq U \).

The grader will carefully consider 1.3.30 and 9 above, so you should write these up more carefully.