Problem 1 (15pts) Quick computations and examples. (There isn’t much work to show for this problem).

(a–5pts) If \( g(x) = \sum_{n=0}^{\infty} \binom{n+3}{n}x^n \) is a generating function for the sequence \( \{h_n\}_{n=0}^{\infty} \), what is a formula for \( h_n \)?

Solution. If \( \sum_{n=0}^{\infty} \binom{n+3}{n}x^n = \sum_{n=0}^{\infty} h_n x^n \) then \( h_n = \binom{n+3}{n} = \frac{(n+3)!}{n!3!} \).

(b–5pts) If \( g^{(e)}(x) = \sum_{n=0}^{\infty} \binom{n+3}{n}x^n \) is an exponential generating function for the sequence \( \{h_n\}_{n=0}^{\infty} \), what is a formula for \( h_n \)?

Solution. If \( \sum_{n=0}^{\infty} \binom{n+3}{n}x^n = \sum_{n=0}^{\infty} \frac{h_n}{n!}x^n \), then \( \frac{h_n}{n!} = \binom{n+3}{n} = \frac{(n+3)!}{n!3!} \) so \( h_n = \frac{(n+3)!}{3!} \).

(c–5pts) Consider the poset \( (\mathcal{P}(\{1\}), \subseteq) \). Compute \( (\delta * \zeta)(\emptyset, \{1\}) \) where \( \delta \) and \( \zeta \) are defined as in the text (in the chapter on Möbius inversion).

Solution. Recall that

\[
\delta(A, B) = \begin{cases} 
1 & \text{if } A = B \\
0 & \text{otherwise}
\end{cases} \quad \text{and} \quad \zeta(A, B) = \begin{cases} 
1 & \text{if } A \subseteq B \\
0 & \text{otherwise}
\end{cases}
\]

So \( \delta(\emptyset, \emptyset) = \zeta(\emptyset, \{1\}) = \zeta(\{1\}, \{1\}) = \zeta(\emptyset, \{1\}) = 1 \), and \( \delta(\emptyset, \{1\}) = 0 \).

Thus

\[
(\delta * \zeta)(\emptyset, \{1\}) = \sum_{\{z \mid \emptyset \subseteq z \subseteq \{1\}\}} \delta(\emptyset, z)\zeta(z, \{1\}) = \delta(\emptyset, \emptyset)\zeta(\emptyset, \{1\}) + \delta(\emptyset, \{1\})\zeta(\{1\}, \{1\})
\]

\[
= 1 \cdot 1 + 0 \cdot 1 = 1.
\]
Problem 2 (10pts) Compute the number of positive integer solutions to the equation \( x_1 + x_2 + x_3 = 20 \) where \( 1 \leq x_1, x_2, x_3 \leq 10 \).

Solution. First make the change of variable \( y_i = x_i - 1 \) for \( i = 1, 2, 3 \). Then it is equivalent to find all positive integer solutions to \( y_1 + y_2 + y_3 = 17 \) with \( 0 \leq y_1, y_2, y_3 \leq 9 \). Let \( S \) be the collection of all 3-tuples of positive integers \((y_1, y_2, y_3)\) such that \( y_1 + y_2 + y_3 = 17 \), and for each \( i = 1, 2, 3 \), let \( A_i \) be all 3-tuples \((y_1, y_2, y_3)\) in \( S \) such that \( y_i \geq 10 \). The number of positive integer solutions to \( y_1 + y_2 + y_3 = 17 \) with \( 0 \leq y_1, y_2, y_3 \leq 9 \) is thus (by inclusion/exclusion)

\[
|A_1 \cap A_2 \cap A_3| = |S| - \sum_{i=1}^{3} |A_i| + \sum_{i \neq j} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3|.
\]

We count the various pieces. Using our formula from the text, \( |S| = \binom{17 + 2}{2} \). To compute \( |A_i| \) for fixed \( i \), we make a second change of variable \( z_i = y_i - 10 \) and \( z_j = y_j \) for \( j \neq i \). Then the system in question becomes \( z_1 + z_2 + z_3 = 7 \), and the number of positive integer solutions is \( \binom{7 + 2}{2} \) by the same formula used before.

Note that \( A_i \cap A_j = \emptyset \) for all \( i \neq j \) and \( A_1 \cap A_2 \cap A_3 = \emptyset \) since if two or three variables are at least 10, then the sum must exceed 17.

Thus the total number of solutions is \( |S| - \sum_{i=1}^{3} |A_i| = \binom{17 + 2}{2} - 3 \binom{7 + 2}{2} \).

Problem 3 (10pts) Determine the number of permutations of \( \{1, 2, 3, 4\} \) such that at least one odd integer is in its natural position. (Recall that given a permutation \( i_1 i_2 i_3 i_4 \) of \( \{1, 2, 3, 4\} \), \( j \) is in its natural position if \( i_j = j \).

Solution. There are only two possibilities if both 1 and 3 end up in their natural position, namely 1234 and 1432.

If only one odd number ends in its natural position, then we have \( \binom{2}{1} \) choices of which odd stays fixed, 2 choices of where to put the other odd element (since it cannot go to its natural position), and 2 ways to place the remaining two elements (since there are exactly 2 open spaces remaining). Thus the total number of permutations of the desired type is \( 2 \cdot 2 \cdot 2 = 10 \).

Problem 4 (10pts) A certain chalk manufacturing company makes four different colors of chalk: white, yellow, orange, and purple. Let \( h_n \) denote the number of ways to put \( n \) pieces of chalk in a box if in each box there are a multiple of 3 number of white sticks of chalk, and even number of yellow sticks of chalk, at most two orange sticks of chalk, and at most one stick of purple chalk. Determine the generating function for \( \{h_n\} \) and use it to find a formula for \( h_n \). (Recall that \( 1 + x + x^2 = \frac{1 - x^3}{1 - x} \)).

Solution. We need the coefficient of \( x^n \) to be a choice of \( c_1, c_2, c_3, c_4 \geq 0 \) such that \( c_1 \) is a multiple of 3, \( c_2 \) is even, \( c_3 \leq 2 \) and \( c_4 \leq 1 \). This is exactly the coefficient of \( x^n \) in the product

\[
g(x) = \left( \sum_{n=0}^{\infty} x^{3n} \right) \left( \sum_{n=0}^{\infty} x^{2n} \right) (1 + x + x^2)(1 + x) = \frac{1}{1 - x^3} \frac{1}{1 - x^2} \frac{1 - x^3}{1 - x} (1 + x) = \frac{1}{(1 + x)(1 - x)} \frac{1}{1 - x} (1 + x)
\]

We count the various pieces. Using our formula from the text, \( |S| = \binom{17 + 2}{2} \).
\[ \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \binom{n+1}{1} x^n = \sum_{n=0}^{\infty} (n+1)x^n, \]

where we used the formula \( \frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n \). So \( h_n = n + 1 \) for all \( n \geq 0 \). \( \square \)

**Problem 5** (15pts) Let \( S \) be the multiset \( \{ \infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4 \} \).

(a–5pts) Give the exponential generating function for the sequence \( \{ h_n \} \) where for all \( n \geq 0 \) \( h_n \) is the number of \( n \) permutations of elements of \( S \). Reduce your answer until it does not include any infinite sums.

**Solution.** According to our theorem in the text, \[
g^{(e)}(x) = f_\infty(x) f_\infty(x) f_\infty(x) f_\infty(x) = \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right)^4 = (e^x)^4 = e^{4x}. \]

\( \square \)

(b–10pts) Give the exponential generating function for the sequence \( \{ h_n \} \) where \( h_n \) is the number of \( n \) permutations of \( S \) such that \( a_1 \) occurs at least twice, \( a_2 \) occurs an even number of times, and \( a_3 \) occurs at most 10 times (there is no restriction on the number of times \( a_4 \) occurs). Reduce your answer until it does not include any infinite sums. (You do not need to justify that the power series you write down is the correct one.)

**Solution.** Here we have
\[
g^{(e)}(x) = \left( \sum_{n=2}^{\infty} x^n \right) \left( \sum_{n=0}^{\infty} x^{2n} \right) \left( \sum_{n=0}^{10} x^n \right) \left( \sum_{n=0}^{\infty} x^n \right)
= \left( \sum_{n=0}^{\infty} x^n - 1 - x \right) \frac{1}{2} \left( \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (-1)^n x^n \right) \left( \sum_{n=0}^{10} x^n \right) \left( \sum_{n=0}^{\infty} x^n \right)
= (e^x - 1 - x) \frac{1}{2} (e^x + e^{-x}) \left( \sum_{n=0}^{10} x^n \right) (e^x). \]

\( \square \)

**Problem 6** (10pts) Solve the recurrence relation \( h_n = -h_{n-1} + 2h_{n-2}, \ (n \geq 2) \) with initial values \( h_0 = 1 \) and \( h_1 = 2 \).

**Solution.** We have two methods. First, by a theorem we know that there are \( c_1, c_2 \in \mathbb{C} \) such that \( h_n = c_1 q_1^n + c_2 q_2^n \) provided that \( q_1, q_2 \) are the distinct roots of the polynomial \( x^2 + x - 2 = 0 \). In this case \( q_1^2 + q_2 - 2 = 0 \) if and only if \( q = 1, -2 \), so we are looking for \( c_1, c_2 \in \mathbb{C} \) such that \( h_n = c_1 \cdot 1^n + c_2 \cdot (-2)^n \), or in particular that
\[
1 = h_0 = c_1 \cdot 1^0 + c_2 \cdot (-2)^0
2 = h_1 = c_1 \cdot 1^1 + c_2 \cdot (-2)^1,
\]

\[ \begin{align*}
1 &= h_0 = c_1 \cdot 1^0 + c_2 \cdot (-2)^0 \\
2 &= h_1 = c_1 \cdot 1^1 + c_2 \cdot (-2)^1,
\end{align*} \]
that is
\[ 1 = c_1 + c_2 \]
\[ 2 = c_1 - 2c_2. \]

or
\[ 2 = 2c_1 + 2c_2 \]
\[ 2 = c_1 - 2c_2. \]

So \( 4 = 3c_1 \) or \( c_1 = 4/3 \) whence \( c_2 - 1/3 \) and the solution is \( h_n = \frac{4}{3} - \frac{1}{3}(-2)^n \).

To demonstrate a second method using generating functions, we note that
\[
\begin{align*}
g(x) &= h_0 + h_1 x + h_2 x^2 + \cdots h_n x^n + \cdots \\
xg(x) &= h_0 x + h_1 x^2 + h_2 x^3 + \cdots h_n x^{n+1} + \cdots \\
-2x^2 g(x) &= -2h_0 x^2 + -2h_1 x^3 + -2h_2 x^4 + \cdots -2h_n x^{n+2} + \cdots \\
(1 + x - 2x^2)g(x) &= h_0 + (h_1 + h_0) x + (h_2 + h_1 - 2h_0) x^2 + \cdots + (h_n + h_{n-1} - 2h_{n-2}) x^n + \cdots
\end{align*}
\]

So
\[
(1 + x - 2x^2)g(x) = 1 + 3x + 0 \cdot x^2 + \cdots = 1 + 3x,
\]
that is
\[
g(x) = \frac{1 + 3x}{1 + x - 2x^2} = \frac{1 + 3x}{(1 - x)(1 + 2x)} = \frac{4/3}{1 - x} + \frac{-1/3}{1 + 2x} = \frac{4}{3} \sum_{n=0}^{\infty} x^n - \frac{1}{3} \sum_{n=0}^{\infty} (-2)^n x^n = \sum_{n=0}^{\infty} \left( \frac{4}{3} - \frac{1}{3}(-2)^n \right) x^n.
\]

So \( h_n = \frac{4}{3} - \frac{1}{3}(-2)^n \) as before.

If you really want to see the partial fraction decomposition . . . we begin with
\[
\frac{1 + 3x}{(1 - x)(1 + 2x)} = \frac{A}{1 - x} + \frac{B}{1 + 2x}.
\]

Clearing denominators yields
\[
1 + 3x = A(1 + 2x) + B(1 - x) = (A + B) + (2A - B)x
\]
and equating coefficients gives
\[
1 = A + B
\]
\[
3 = 2A - B.
\]

Adding the two equations gives \( 4 = 3A \), so \( A = 4/3 \) whence \( B = 1 - A = -1/3 \), so
\[
\frac{1 + 3x}{(1 - x)(1 + 2x)} = \frac{4/3}{1 - x} + \frac{-1/3}{1 + 2x}.
\]