

Solution to 12.1.64,
(and some comments about the Texas-Alabama game).

Compute $\lim_{n \rightarrow \infty} a_n$ if $\{a_n\}$ is the sequence defined recursively as $a_1 = 2$ and, for $n \geq 1$,
$$a_{n+1} = \frac{1}{3 - a_n}.$$

The outline:

1. Show $a_n \leq 2$ for all $n \in \mathbb{N}$
2. Show $a_n \geq \frac{3 - \sqrt{5}}{2}$ for all $n \in \mathbb{N}$
3. Use 1. and 2. to show that $\{a_n\}$ is decreasing.
4. Use 1., 2., and 3. to show that $\{a_n\}$ converges
5. Use 4. to solve for $\lim_{n \rightarrow \infty} a_n$.

We first show that $a_n \leq 2$ for all $n \in \mathbb{N}$. If not, there must be a “first time” that the sequence goes above 2. Let m be the smallest subscript such that $a_m > 2$ (and then, since this is the first occurrence, it must be that $a_1, \dots, a_{m-1} \leq 2$). Note that since $a_1 = 2$, we know that $m > 1$.

Since $a_{m-1} \leq 2$, we see that $3 - a_{m-1} > 0$, and thus

$$2 < a_m = \frac{1}{3 - a_{m-1}}$$

can be cross multiplied to yield

$$2(3 - a_{m-1}) < 1$$

and then manipulated to yield

$$6 - 2a_{m-1} < 1$$

and again

$$6 - 1 < 2a_{m-1}$$

and again

$$\frac{5}{2} < a_{m-1}.$$

But $2 < \frac{5}{2}$, so

$$2 < \frac{5}{2} < a_{m-1}.$$

Alas! We already know that $a_{m-1} \leq 2$. Egads!

$$2 < a_{m-1} \leq 2.$$

This is a contradiction. We forced this impossibility by seeing what would happen if $a_n > 2$ ever occurred. We now conclude that it never does.

Next we show that $a_n \geq \frac{3 - \sqrt{5}}{2}$ for all n (the reason for this will become clear in a bit). (*Begin cut and paste from above, as the argument is so similar*). If not there must be a “first time” that the sequence goes below $\frac{3 - \sqrt{5}}{2}$. Let m be the smallest subscript such that $a_m < \frac{3 - \sqrt{5}}{2}$ (and then, since this is the first occurrence, it must be that $a_1, \dots, a_{m-1} \geq \frac{3 - \sqrt{5}}{2}$). Note that $m > 1$.

Now

$$\frac{3 - \sqrt{5}}{2} > a_m = \frac{1}{3 - a_{m-1}}$$

can be cross multiplied to yield

$$(3 - a_{m-1})(3 - \sqrt{5}) > 2$$

and then manipulated to yield

$$9 - 3\sqrt{5} - a_{m-1}(3 - \sqrt{5}) > 2$$

and again

$$-a_{m-1}(3 - \sqrt{5}) > 2 - (9 - 3\sqrt{5})$$

and again

$$-a_{m-1}(3 - \sqrt{5}) > -7 + 3\sqrt{5}$$

and again

$$a_{m-1}(3 - \sqrt{5}) < 7 - 3\sqrt{5}$$

and again

$$a_{m-1} < \frac{7 - 3\sqrt{5}}{3 - \sqrt{5}}$$

and again

$$\begin{aligned} a_{m-1} &< \left(\frac{7 - 3\sqrt{5}}{3 - \sqrt{5}} \right) \left(\frac{3 + \sqrt{5}}{3 + \sqrt{5}} \right) = \frac{(7 - 3\sqrt{5})(3 + \sqrt{5})}{(3 - \sqrt{5})(3 + \sqrt{5})} \\ &= \frac{21 + 7\sqrt{5} - 9\sqrt{5} - 15}{9 - 5} = \frac{6 - 2\sqrt{5}}{4} = \frac{3 - \sqrt{5}}{2}. \end{aligned}$$

Simplifying, this says

$$a_{m-1} < \frac{3 - \sqrt{5}}{2}.$$

Alas! We already know that

$$\frac{3 - \sqrt{5}}{2} \leq a_{m-1},$$

and so we have

$$\frac{3 - \sqrt{5}}{2} \leq a_{m-1} < \frac{3 - \sqrt{5}}{2}.$$

This is a contradiction. We forced this impossibility by seeing what would happen if $a_n < \frac{3 - \sqrt{5}}{2}$ ever occurred. We now concluded that it never does.

Now we show that $\{a_n\}$ is decreasing. We can do this by showing that $a_n - a_{n+1} = a_n - \frac{1}{3 - a_n}$ is always positive. Let $f(x) = x - \frac{1}{3 - x}$. Then it is enough to show that $f(a_n)$ is always positive. Of course, we showed above $\frac{3 - \sqrt{5}}{2} \leq a_n \leq 2$, so we will be done if we show that $f(x)$ is always positive on $\left[\frac{3 - \sqrt{5}}{2}, 2\right]$. Note that $f(x)$ is continuous on $\left[\frac{3 - \sqrt{5}}{2}, 2\right]$ and

$$\begin{aligned} f\left(\frac{3 - \sqrt{5}}{2}\right) &= \frac{3 - \sqrt{5}}{2} - \frac{1}{3 - \frac{3 - \sqrt{5}}{2}} = \frac{3 - \sqrt{5}}{2} - \frac{1}{\frac{6 - 3 + \sqrt{5}}{2}} \\ &= \frac{3 - \sqrt{5}}{2} - \frac{2}{3 + \sqrt{5}} = \left(\frac{3 - \sqrt{5}}{2}\right) \left(\frac{3 + \sqrt{5}}{3 + \sqrt{5}}\right) - \frac{4}{2(3 + \sqrt{5})} \\ &= \frac{9 - 5}{2(3 + \sqrt{5})} - \frac{4}{2(3 + \sqrt{5})} = 0. \end{aligned}$$

So, if we can show that $f'(x) \geq 0$ on $[0, 2]$, then $f'(x)$ will be positive on $\left[\frac{3 - \sqrt{5}}{2}, 2\right]$, and hence $f(x)$ will be positive on $\left[\frac{3 - \sqrt{5}}{2}, 2\right]$ (because it is positive at the left endpoint, has positive slope, and is continuous on the interval).

Of course, $f'(x) = 1 - \frac{1}{(3 - x)^2}$. We need to show that this is positive. Here goes. We know that

$$0 \leq x \leq 2$$

so

$$0 \geq -x \geq -2$$

so

$$3 \geq 3 - x \geq 3 - 2$$

so

$$3 \geq 3 - x \geq 1$$

so

$$9 \geq (3 - x)^2 \geq 1$$

so

$$1 \geq \frac{1}{(3 - x)^2} \geq \frac{1}{9}$$

so

$$-1 \leq -\frac{1}{(3 - x)^2} \leq -\frac{1}{9}$$

so

$$1 - 1 \leq 1 - \frac{1}{(3 - x)^2} \leq 1 - \frac{1}{9}$$

so

$$0 \leq f'(x) \leq \frac{8}{9}.$$

It's positive. And there was much rejoicing.

Ok, now we have that $\{a_n\}$ is a monotone, bounded sequence, so it converges by our theorem. Let's say that $a_n \rightarrow a$. Then

$$a = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \left(\frac{1}{3 - a_n} \right) = \frac{1}{3 - \lim_{n \rightarrow \infty} a_n} = \frac{1}{3 - a}.$$

So $a = \frac{1}{3 - a}$ or $a(3 - a) = 1$ or $a^2 - 3a + 1 = 0$ or (using the quadratic formula, known to the ancient Babylonians), $a = \frac{3 \pm \sqrt{5}}{2}$. Since we know that $a_n \leq 2$ for all n , it must be that $a = \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} 2 = 2$, and thus $a_n \rightarrow \frac{3 - \sqrt{5}}{2}$.

Ok, so that was a bit... well... ummm ... ok, I'll say it ... long. Partly that is because I was abusive in the level of detail I used (hopefully making it as clear as possible). What can you get from it? Read the answer, and think about the flavor of it; then let it go.

Onto the game. I was happy (elated? ecstatic? words fail me) that Texas lost (not that Alabama won; such is my dislike of Texas), but sad about McCoy. I wanted to see the best go against the best (which would have been Florida versus Alabama, but I digress ...) and having the star QB on the sidelines was no fun. Plus, I don't want people hurt; it is just a game.