Problem 1. (10pts) Give both the vector equation and parametric equations of the line from the point (1, 2, 3) to the point (3, 4, 5).

Solution. The vector from (1, 2, 3) to (2, 3, 4) is \( (3-1, 4-2, 5-3) = (2, 2, 2) \) (subtracting the points in the usual way). So, using the usual formula, a vector equation for the line through these two points is \( \vec{T}(t) = (1, 2, 3) + t(2, 2, 2) = (1+2t, 2+2t, 3+2t) \).
We can then read the parametric equations from the vector equation obtaining:

\[
\begin{align*}
x(t) &= 1 + 2t \\
y(t) &= 2 + 2t \\
z(t) &= 3 + 2t
\end{align*}
\]

□

Problem 2. (10pts) Decide if the slope of the tangent to \( x(t), y(t) \) is positive, negative, or zero at \( t = 1/2 \) given the graphs of \( x = x(t) \) and \( y = y(t) \) below:

![Graphs of x(t) and y(t)](image_url)

Solution. We know that \( \frac{dy}{dx} \bigg|_{t=1/2} = \frac{y'(1/2)}{x'(1/2)} \). It is clear, by simply considering the graphs, that \( x'(1/2) < 0 \) (the slope of the tangent to \( x(t) \) is negative at \( t = 1/2 \)), while \( y'(1/2) > 0 \) (the slope of the tangent to \( y(t) \) is positive at \( t = 1/2 \)). We conclude that \( \frac{dy}{dx} \bigg|_{t=1/2} < 0 \).

□

Problem 3. (10pts) Find the area which is inside both \( r = 1 + \cos \theta \) and \( r = \sin \theta \). (A side note is that \( 1 + \cos \theta = \sin \theta \) at \( \theta = \pi/2 \) among other values).

Solution. Using pictures of \( r = \sin \theta \) and \( r = 1 + \cos \theta \) in the \((\theta, r)\) plane,
we can develop a graph of both in the \((x, y)\)-plane by noting that, as \(\theta\) swings from 0 to \(\pi\), \(r = \sin \theta\) goes from 0 to 1 and back to 0, while \(r = 1 + \cos \theta\) goes from 2 down to 0, and that for \(\pi \leq \theta \leq 2\pi\), the \(\sin \theta\) function takes negative values (thus retracing the circle it made for \(0 \leq \theta \leq \pi\), while the values of \(1 + \cos \theta\) are still positive, and hence live below the \(x\)-axis.

We also note that, as per the hint given above, the point of intersection of the two graphs comes at \(\theta = \pi/2\). Since we are interested in the area inside both curves, we compute the area inside \(\sin \theta\) for \(0 \leq \theta \leq \pi/2\) and add this to the area inside \(1 + \cos \theta\) for \(\pi/2 \leq \theta \leq \pi\). Using the usual formula for area inside a polar curve, we obtain:

\[
\int_0^{\pi/2} \frac{(\sin \theta)^2}{2} d\theta + \int_{\pi/2}^{\pi} \frac{(1 + \cos \theta)^2}{2} d\theta = \int_0^{\pi/2} \frac{1 - \cos 2\theta}{4} d\theta + \frac{1}{2} \int_{\pi/2}^{\pi} (1 + 2 \cos \theta + \cos 2\theta) d\theta
\]

\[
= \left(\frac{\theta}{4} - \frac{\sin 2\theta}{8}\right) \bigg|_0^{\pi/2} + \frac{1}{2} \int_{\pi/2}^{\pi} \left(1 + \sin \theta + \sin \frac{2\theta}{2}\right) d\theta = \frac{\pi}{8} + \frac{1}{2} \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) \bigg|_{\pi/2}^{\pi}
\]

\[
= \frac{\pi}{8} + \left(\frac{3\theta}{4} + \sin \frac{2\theta}{2}\right) \bigg|_{\pi/2}^{\pi} = \frac{\pi}{8} + \left(\frac{3\pi}{4} + 1\right) = \frac{\pi}{2} - 1.
\]

\(\blacksquare\)

Problem 4. \((15\text{pts})\) Consider the vectors \(\vec{v} = (1, 2, 3)\) and \(\vec{u} = (1, 0, -2)\).

\((a-5\text{pts})\) Compute the projection of \(\vec{v}\) onto the vector \(\vec{u}\).

\(\text{Solution.}\) Following the usual formula, we have

\[
\text{proj}_u \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|u|^2} \vec{u} = \frac{(1, 0, -2) \cdot (1, 2, 3)}{(12 + 0^2 + (-2)^2)}(1, 0, -2) = \frac{1 - 6}{\sqrt{5}}(1, 0, -2) = \left\langle \frac{-5}{\sqrt{5}}, 0, \frac{-10}{\sqrt{5}} \right\rangle.
\]

\(\blacksquare\)

\((b-5\text{pts})\) Give a vector which is parallel to \(\vec{u}\), but has the same length as \(\vec{v}\).

\(\text{Solution.}\) From the previous problem we know that \(|\vec{u}| = \sqrt{5}\) while \(|\vec{v}| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{14}\). Thus a unit vector parallel to \(\vec{u}\) is \(\frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{5}}(1, 0, -2) = \left\langle \frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}} \right\rangle\), and hence a vector of length \(\sqrt{14}\) which is parallel to \(\vec{u}\) is

\[
\sqrt{14} \left\langle \frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}} \right\rangle = \left\langle \frac{\sqrt{14}}{\sqrt{5}}, 0, -\frac{2\sqrt{14}}{\sqrt{5}} \right\rangle.
\]

That this vector is parallel to \(\vec{u}\) follows because it is a scalar times \(\vec{u}\). That it has length \(\sqrt{14}\) is verified by the vector rules, or by a quick check:

\[
\left| \left\langle \frac{\sqrt{14}}{\sqrt{5}}, 0, -\frac{2\sqrt{14}}{\sqrt{5}} \right\rangle \right| = \sqrt{\left(\frac{\sqrt{14}}{\sqrt{5}}\right)^2 + (0)^2 + \left(\frac{-2\sqrt{14}}{\sqrt{5}}\right)^2} = \sqrt{\frac{14}{5} + 4 \frac{14}{5}} = \sqrt{\frac{5 \cdot 14}{5}} = \sqrt{14}
\]
as required.

\( c \)-5pts Give a vector which is perpendicular to \( \vec{v} \), but not perpendicular to \( \vec{u} \).

Solution. We can’t use \( \vec{v} \times \vec{u} \) since this is perpendicular to both \( \vec{v} \) and \( \vec{u} \). How about using a random vector, like \( \langle 1, 0, 0 \rangle \)? Then

\[
\begin{vmatrix}
1 & 2 & 3 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{vmatrix} = (0 - 0, -(0 - 3), 0 - 2) = (0, 3, -2)
\]

is perpendicular to \( \vec{v} \) (by the properties of the cross product), but not perpendicular to \( \vec{u} \) since \( \langle 0, 3, -2 \rangle \cdot \langle 0, 1, -2 \rangle = 0 + 0 + 4 \neq 0 \)

\( \square \)

Problem 5. \( 20 \)pts Consider the planes \( 2x + 2y - z = 4 \) and \( 3x - 2y + z = 2 \).

(a)-10pts Suppose that \( \theta \) is the angle between these two planes. Compute \( \sin \theta \).

Solution. The normal vectors to these two planes are \( \langle 2, 2, -1 \rangle \) and \( \langle 3, -2, 1 \rangle \) respectively, and we know that the angle between the two planes is the same as the angle between their normals. So, using the formula for cross products,

\[
\sin \theta = \frac{|\langle 2, 2, -1 \rangle \times \langle 3, -2, 1 \rangle|}{|\langle 2, 2, -1 \rangle||\langle 3, -2, 1 \rangle|} = \frac{5\sqrt{5}}{3\sqrt{14}}.
\]

Off to the side, we have computed that

\[
\langle 2, 2, -1 \rangle \times \langle 3, -2, 1 \rangle = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
2 & 2 & -1 \\
3 & -2 & 1
\end{vmatrix} = (2 - 2, -(2 + 3), -4 - 6) = (0, -5, -10),
\]

and that

\[
|\langle 0, -5, -10 \rangle| = \sqrt{125} = 5\sqrt{5},
\]

and that

\[
|\langle 2, -2, 1 \rangle| = \sqrt{9} = 3,
\]

and finally that

\[
|\langle 3, -2, 1 \rangle| = \sqrt{14}.
\]

\( \square \)

(b)-10pts Give the equation of the line formed by the intersection of these two planes.

Solution. We know that the direction of the line in question is given by \( \langle 0, -5, -10 \rangle \) since this is a vector which is perpendicular to the normals of both planes (and hence lies in both planes). Adding the first equation to the second yields the system \( (2x + 2y - z) + (3x - 2y + z) = 4 + 2, \) or \( 5x = 6 \) and hence \( x = 6/5 \). Then \( 2x + 2y - z = 4 \) gives \( 12/5 + 2y - z = 4 \) or \( 2y - z = 8/5 \). Taking \( y = 0 \) yields \( z = -8/5 \). So \( (6/5, 0, -8/5) \) is on the intersection of both planes, and then using the usual procedure, the plane in question is

\[
0(x - 6/5) - 5(y - 0) - 10(z + 8/5) = 0.
\]

\( \square \)
Problem 6. (12pts) Match the following graphs with their equations. You do not need to show any work for this problem.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A) ( r = \theta/2 )</td>
<td>IV</td>
</tr>
<tr>
<td>(B) ( r = \text{arctan}(2\theta) )</td>
<td>III</td>
</tr>
<tr>
<td>(C) ( r = \sin(\theta)\sin(2\theta) )</td>
<td>II</td>
</tr>
<tr>
<td>(D) ( r = \sin(2\sin(\theta)) )</td>
<td>I</td>
</tr>
</tbody>
</table>