Calculus III

Math 143 Winter 2010

Professor Ben Richert

Exam 2

Solutions

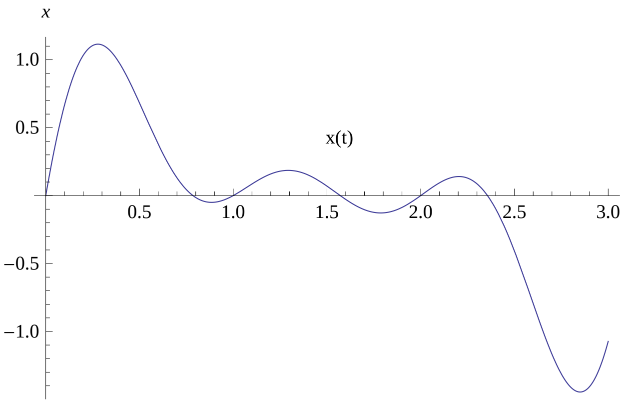
Problem 1. *(10pts) Give both the vector equation and parametric equations of the line from the point* $(1,2,3)$ *to the point* $(3,4,5)$.

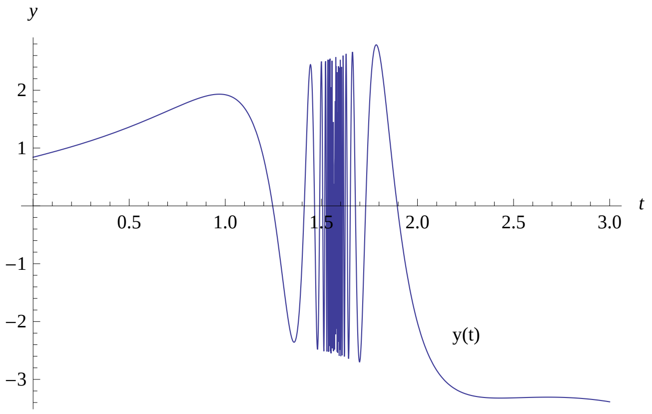
Solution. The vector from $(1,2,3)$ to $(2,3,4)$ is $\langle 3-1, 4-2, 5-3\rangle = \langle 2,2,2\rangle$ (subtracting the points in the usual way). So, using the usual formula, a vector equation for the line through these two points is $\overrightarrow{r}(t) = \langle 1,2,3\rangle + t\langle 2,2,2\rangle = \langle 1+2t, 2+2t, 3+2t\rangle$. We can then read the parametric equations from the vector equation obtaining:

$$\begin{aligned} x(t) &= 1+2t \\ y(t) &= 2+2t \\ z(t) &= 3+2t \end{aligned}$$

$\square$

Problem 2. *(10pts) Decide if the slope of the tangent to* $x(t)$, $y(t)$ *is positive, negative, or zero at* $t = 1/2$ *given the graphs of* $x = x(t)$ *and* $y = y(t)$ *below:*

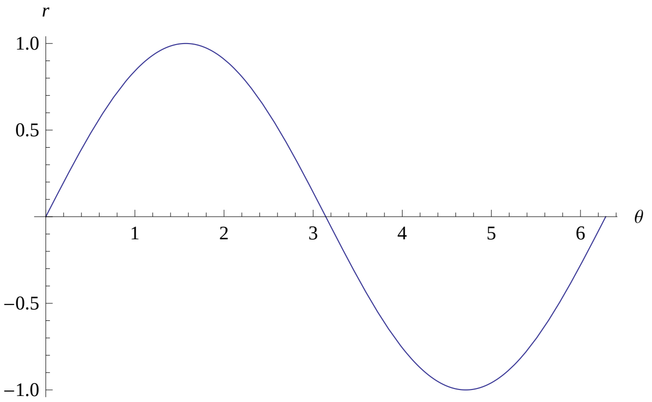


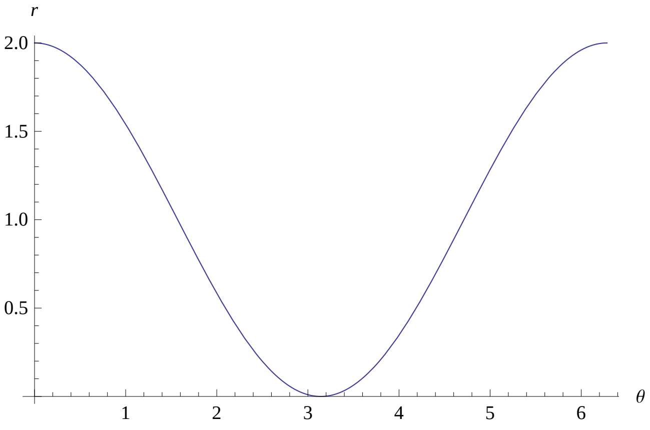


Solution. We know that $\left.\dfrac{dy}{dx}\right|_{t=1/2} = \dfrac{y'(1/2)}{x'(1/2)}$. It is clear, by simply considering the graphs, that $x'(1/2) < 0$ (the slope of the tangent to $x(t)$ is negative at $t = 1/2$), while $y'(1/2) > 0$ (the slope of the tangent to $y(t)$ is positive at $t = 1/2$). We conclude that $\left.\dfrac{dy}{dx}\right|_{t=1/2} < 0$. $\square$

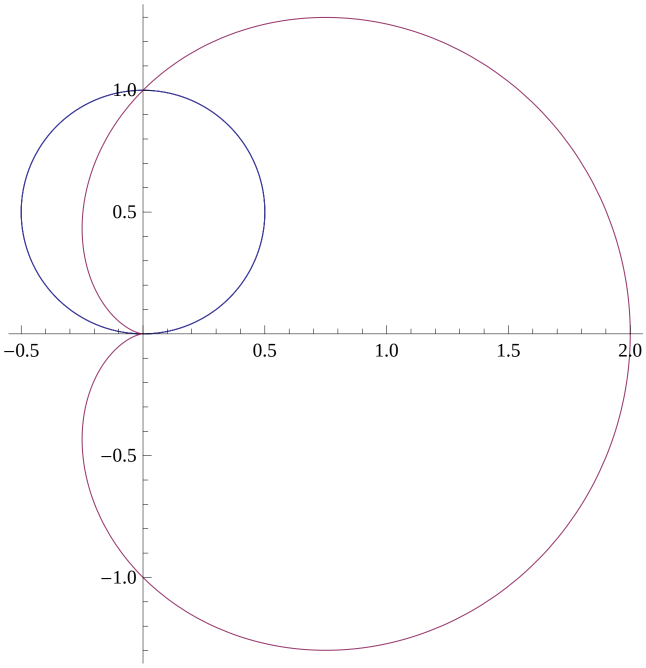
Problem 3. *(10pts) Find the area which is inside both* $r = 1+\cos\theta$ *and* $r = \sin\theta$. *(A side note is that* $1+\cos\theta = \sin\theta$ *at* $\theta = \pi/2$ *among other values).*

Solution. Using pictures of $r = \sin\theta$ and $r = 1+\cos\theta$ in the (θ, r) plane,





we can develop a graph of both in the (x, y)-plane by noting that, as θ swings from 0 to π, $r = \sin\theta$ goes from 0 to 1 and back to 0, while $r = 1 + \cos\theta$ goes from 2 down to 0, and that for $\pi \le \theta \le 2\pi$, the $\sin\theta$ function takes negative values (thus retracing the circle it made for $0 \le \theta \le \pi$, while the values of $1 + \cos\theta$ are still positive, and hence live below the x-axis.



We also note that, as per the hint given above, the point of intersection of the two graphs comes at $\theta = \pi/2$. Since we are interested in the area inside both curves, we compute the area inside $\sin\theta$ for $0 \le \theta \le \pi/2$ and add this to the area inside $1 + \cos\theta$ for $\pi/2 \le \theta \le \pi$. Using the usual formula for area inside a polar curve, we obtain:

$$\int_0^{\pi/2} \frac{(\sin\theta)^2}{2}\,d\theta + \int_{\pi/2}^{\pi} \frac{(1+\cos\theta)^2}{2}\,d\theta = \int_0^{\pi/2} \frac{1-\cos 2\theta}{4}\,d\theta + \frac{1}{2}\int_{\pi/2}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \left(\frac{\theta}{4} - \frac{\sin 2\theta}{8}\right)\Bigg|_0^{\pi/2} + \frac{1}{2}\int_{\pi/2}^{\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta = \frac{\pi}{8} + \frac{1}{2}\left(\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Bigg|_{\pi/2}^{\pi}$$

$$= \frac{\pi}{8} + \left(\frac{3\theta}{4} + \sin\theta + \frac{\sin 2\theta}{8}\right)\Bigg|_{\pi/2}^{\pi} = \frac{\pi}{8} + \left(\frac{3\pi}{4}\right) - \left(\frac{3\pi}{8} + 1\right) = \frac{\pi}{2} - 1.$$

$\square$

Problem 4. *(15pts) Consider the vectors* $\overrightarrow{v} = \langle 1, 2, 3\rangle$ *and* $\overrightarrow{u} = \langle 1, 0, -2\rangle$.

(a–5pts) Compute the projection of $\overrightarrow{v}$ *onto the vector* $\overrightarrow{u}$.

Solution. Following the usual formula, we have

$$\operatorname{proj}_{\overrightarrow{u}}\overrightarrow{v} = \frac{\overrightarrow{u}\cdot\overrightarrow{v}}{|\overrightarrow{u}|^2}\overrightarrow{u} = \frac{\langle 1,0,-2\rangle\cdot\langle 1,2,3\rangle}{(\sqrt{1^2+0^2+(-2)^2})^2}\langle 1,0,-2\rangle = \frac{1-6}{\sqrt{5}}\langle 1,0,-2\rangle = \left\langle \frac{-5}{\sqrt{5}}, 0, \frac{10}{\sqrt{5}}\right\rangle.$$

$\square$

(b–5pts) Give a vector which is parallel to $\overrightarrow{u}$*, but has the same length as* $\overrightarrow{v}$.

Solution. From the previous problem we know that $|\overrightarrow{u}| = \sqrt{5}$ while $|\overrightarrow{v}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$. Thus a unit vector parallel to $\overrightarrow{u}$ is $\frac{\overrightarrow{u}}{|\overrightarrow{u}|} = \frac{1}{\sqrt{5}}\langle 1,0,-2\rangle = \left\langle \frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}\right\rangle$, and hence a vector of length $\sqrt{14}$ which is parallel to $\overrightarrow{u}$ is

$$\sqrt{14}\left\langle \frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}\right\rangle = \left\langle \frac{\sqrt{14}}{\sqrt{5}}, 0, \frac{-2\sqrt{14}}{\sqrt{5}}\right\rangle.$$

That this vector is parallel to $\overrightarrow{u}$ follows because it is a scalar times $\overrightarrow{u}$. That it has length $\sqrt{14}$ is verified by the vector rules, or by a quick check:

$$\left|\left\langle \frac{\sqrt{14}}{\sqrt{5}}, 0, \frac{-2\sqrt{14}}{\sqrt{5}}\right\rangle\right| = \sqrt{\left(\frac{\sqrt{14}}{\sqrt{5}}\right)^2 + (0)^2 + \left(\frac{-2\sqrt{14}}{\sqrt{5}}\right)^2} = \sqrt{\frac{14}{5} + 4\frac{14}{5}} = \sqrt{5\frac{14}{5}} = \sqrt{14}$$

as required.

$\square$

(c–5pts) Give a vector which is perpendicular to $\overrightarrow{v}$*, but not perpendicular to* $\overrightarrow{u}$*.*

Solution. We can't use $\overrightarrow{v} \times \overrightarrow{u}$ since this is perpendicular to both $\overrightarrow{v}$ and $\overrightarrow{u}$. How about using a random vector, like $\langle 1, 0, 0 \rangle$? Then

$$\langle 1,2,3\rangle \times \langle 1,0,0\rangle = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0-0, -(0-3), 0-2\rangle = \langle 0, 3, -2\rangle$$

is perpendicular to $\overrightarrow{v}$ (by the properties of the cross product), but not perpendicular to $\overrightarrow{u}$ since

$$\langle 0,3,-2\rangle \cdot \langle 1,0,-2\rangle = 0 + 0 + 4 \neq 0$$

(and two vectors are perpendicular if and only if their dot product is zero). $\square$

Problem 5. *(20pts) Consider the planes* $2x + 2y - z = 4$ *and* $3x - 2y + z = 2$.

(a–10pts) Suppose that θ *is the angle between these two planes. Compute* $\sin\theta$*.*

Solution. The normal vectors to these two planes are $\langle 2, 2, -1\rangle$ and $\langle 3, -2, 1\rangle$ respectively, and we know that the angle between the two planes is the same as the angle between their normals. So, using the formula for cross products,

$$\sin\theta = \frac{|\langle 2,2,-1\rangle \times \langle 3,-2,1\rangle|}{|\langle 2,2,-1\rangle||\langle 3,-2,1\rangle|} = \frac{5\sqrt{5}}{3\sqrt{14}}.$$

Off to the side, we have computed that

$$\langle 2,2,-1\rangle \times \langle 3,-2,1\rangle = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & 2 & -1 \\ 3 & -2 & 1 \end{vmatrix} = \langle 2-2, -(2+3), -4-6\rangle = \langle 0, -5, -10\rangle,$$

and that

$$|\langle 0,-5,-10\rangle| = \sqrt{125} = 5\sqrt{5},$$

and that

$$|\langle 2,2,-1\rangle| = \sqrt{9} = 3,$$

and finally that

$$|\langle 3,-2,1\rangle| = \sqrt{14}.$$

$\square$

(b–10pts) Give the equation of the line formed by the intersection of these two planes.

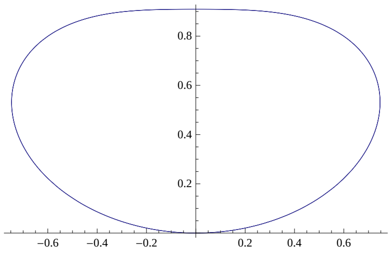
Solution. We know that the direction of the line in question is given by $\langle 0, -5, -10\rangle$ since this is a vector which is perpendicular to the normals of both planes (and hence lies in both planes). Adding the first equation to the second yields the system $(2x + 2y - z) + (3x - 2y + z) = 4 + 2$, or $5x = 6$ and hence $x = 6/5$. Then $2x + 2y - z = 4$ gives $12/5 + 2y - z = 4$ or $2y - z = 8/5$. Taking $y = 0$ yields $z = -8/5$. So $(6/5, 0, -8/5)$ is on the intersection of both planes, and then using the usual procedure, the plane in question is

$$0(x - 6/5) - 5(y - 0) - 10(z + 8/5) = 0.$$

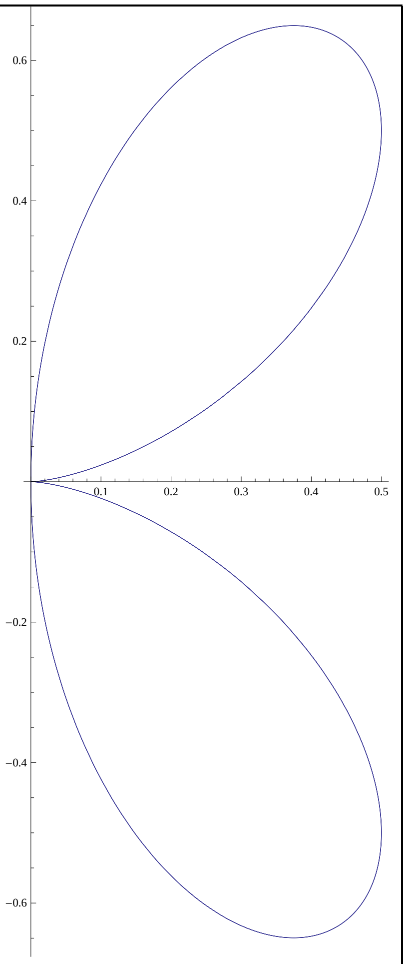
$\square$

Problem 6. *(12pts) Match the following graphs with their equations. You do not need to show any work for this problem.*

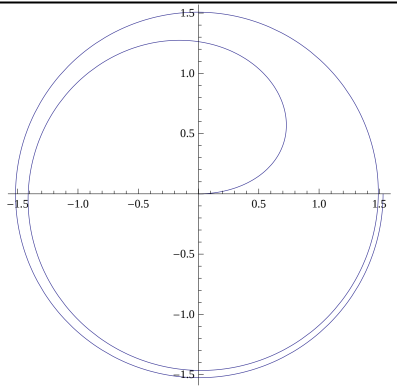
Equation	***Graph***
(A) $r = \theta/2$	*IV*
(B) $r = \arctan(2\theta)$	*III*
(C) $r = \sin(\theta)\sin(2\theta)$	*II*
(D) $r = \sin(2\sin(\theta))$	*I*



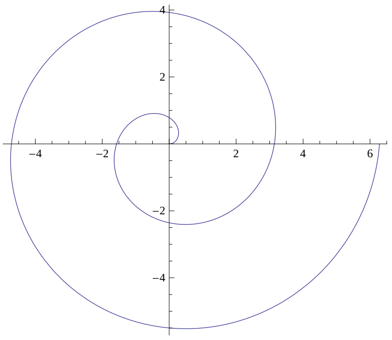
(I)



(II)



(III)



(IV)