

Calculus III
Math 143 Fall 2010
Professor Ben Richert

Exam 1
Solutions

Problem 1. (25pts) Consider the sequence $\left\{ \frac{n}{n+1} \right\}_{n=0}^{\infty}$.

(a-10pts) Does $\left\{ \frac{n}{n+1} \right\}_{n=0}^{\infty}$ converge or diverge?

Solution. In this case the sequence converges because $\lim_{n \rightarrow \infty} \frac{n}{n+1}$ exists. Using the usual nerd trick and the limit laws, we find

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{n}} \left(\frac{n}{n+1} \right) = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1.$$

□

(b-10pts) Does the sequence of partial sums of $\left\{ \frac{n}{n+1} \right\}_{n=0}^{\infty}$ converge or diverge?

Solution. Given a sequence $\{a_n\}_{n=0}^{\infty}$, we use $\sum_{n=0}^{\infty} a_n$ to denote the limit of the sequence of partial sums (when it exists). So this question is asking if $\sum_{n=0}^{\infty} \frac{n}{n+1}$ converges or diverges. Since $\frac{n}{n+1} \not\rightarrow 0$ (as we saw in part (a)), we conclude that $\sum_{n=0}^{\infty} \frac{n}{n+1}$ diverges by the Test for Divergence.

□

(c-5pts) Use your answer in part (b) to show that the power series $\sum_{n=0}^{\infty} \frac{n}{n+1} x^n$ has radius of convergence ≤ 1 .

Solution. If the radius of convergence of this power series was $R > 1$, then the series would converge for all $|x| < R$, and in particular, it would converge for $x = 1$. This cannot be the case, however, since we found in part (b) that $\sum_{n=0}^{\infty} \frac{n}{n+1} = \sum_{n=0}^{\infty} \frac{n}{n+1} 1^n$ diverges. It must be, therefore, that the radius of convergence is ≤ 1 .

□

Problem 2. (10pts) Do exactly one of the following two problems. Indicate clearly which one you want graded.

(a-10pts) Decide whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{6^n + 1}$ converges or diverges.

Solution. It is clear that $\sum_{n=1}^{\infty} \frac{(-1)^n}{6^n + 1}$ alternates. In addition $\lim_{n \rightarrow \infty} \frac{1}{6^n + 1} = 0$ (because the denominator $6^n + 1$ increases without bound as $n \rightarrow \infty$ and therefore $\frac{1}{6^n + 1}$ becomes arbitrarily small), and $\frac{1}{6^n + 1}$ is decreasing (since $6^n + 1$ is an increasing function). Thus $\sum_{n=1}^{\infty} \frac{(-1)^n}{6^n + 1}$ satisfies the conditions for the Alternating Series Test, and hence converges.

□

(b-10pts) Decide whether $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+2} \right)^n$ converges or diverges.

Solution. We use the Root Test. Note that

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{n+1}{2n+2} \right)^n \right|} = \lim_{n \rightarrow \infty} \frac{n+1}{2n+2} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{n}} \left(\frac{n+1}{2n+2} \right) = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{2 + \frac{2}{n}} = \frac{1}{2}.$$

Since the limit is < 1 , we know that $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+2} \right)^n$ converges absolutely by the Root Test. \square

Problem 3. (10pts) Do exactly one of the following two problems. Indicate clearly which one you want graded.

(a-10pts) Decide whether $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converges or diverges.

Solution. Consider the function $f(x) = \frac{1}{x\sqrt{\ln x}}$. We know that $f(x) > 0$ for $x > 1$, $f(x)$ is continuous for $x > 1$ (being made of continuous function which are never zero for $x > 1$), and $f(x)$ is decreasing for $x > 1$ (since both x and $\sqrt{\ln x}$ are increasing functions (the latter being the composition of two increasing functions)). Also, we see that $f(n) = \frac{1}{n\sqrt{\ln n}}$ for all $n \in \mathbb{N}$. Thus, by the integral test, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges if and only if $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx$ diverges.

So here we go:

$$\begin{aligned} \int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x\sqrt{\ln x}} dx = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{1}{u^{1/2}} dx \\ &= \lim_{b \rightarrow \infty} 2u^{1/2} \Big|_{\ln 2}^{\ln b} = \lim_{b \rightarrow \infty} (2\sqrt{\ln b} - 2\sqrt{\ln 2}) = \infty. \end{aligned}$$

We conclude that the integral (and hence also the series) diverges. Here we used the u -substitution $u = \ln x$ (whence $\frac{du}{dx} = \frac{1}{x}$, or lying, $du = \frac{dx}{x}$) and changed the limits of integration to reflect that $u = \ln 2$ when $x = 2$ and $u = \ln b$ when $x = b$. We also used that fact that $\lim_{b \rightarrow \infty} \sqrt{\ln b} = \infty$. \square

(b-10pts) Decide whether $\sum_{n=1}^{\infty} \left(\frac{3n^2+1}{n^2+2} \right) \frac{1}{e^n}$ converges or diverges.

Solution. We compare with $\sum_{n=1}^{\infty} \frac{1}{e^n}$ using the Limit Comparison Test. Note that

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{3n^2+1}{n^2+2} \right) \frac{1}{e^n}}{\frac{1}{e^n}} = \lim_{n \rightarrow \infty} \left(\frac{3n^2+1}{n^2+2} \right) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \left(\frac{3n^2+1}{n^2+2} \right) = \lim_{n \rightarrow \infty} \frac{3 + \frac{1}{n^2}}{1 + \frac{2}{n^2}} = 3 > 0,$$

so by the Limit Comparison Test, both $\sum_{n=1}^{\infty} \left(\frac{3n^2+1}{n^2+2} \right) \frac{1}{e^n}$ and $\sum_{n=1}^{\infty} \frac{1}{e^n}$ converge or both diverge. But

$$\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \frac{1}{e} \left(\frac{1}{e} \right)^{n-1}$$

is a geometric series with $r = \frac{1}{e} < 1$ and is hence convergent by the Geometric Series Test. We conclude that

$\sum_{n=1}^{\infty} \left(\frac{3n^2+1}{n^2+2} \right) \frac{1}{e^n}$ converges as well. \square

Problem 4. (10pts) Compute $T_3(x)$, the Taylor polynomial of degree 3 for the function $f(x) = \sin(x) \cos(x)$.

Solution. The formula for the Taylor polynomial of degree 3 (centered at $x = 0$) is

$$T_3(x) = \sum_{n=0}^3 \frac{f^{(n)}(0)}{n!} x^n = f^{(0)}(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2} x^2 + \frac{f^{(3)}(0)}{3!} x^3.$$

So, plugging and chugging, we have:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x) \cos(x)$	0
1	$-\sin^2(x) + \cos^2(x)$	1
1	$-4 \sin(x) \cos(x)$	0
1	$4 \sin^2(x) - 4 \cos^2(x)$	-4

giving the polynomial

$$T_3(x) = 0 + x + \frac{0}{2}x^2 + \frac{-4}{6}x^3 = x - \frac{2}{3}x^3.$$

□

Problem 5. (10pts) Do exactly one of the following two problems. Indicate clearly which one you want graded.

(a-10pts) Estimate $\int_0^{1/2} \arctan x \, dx$ with error $\leq \frac{1}{(9)(10)2^{10}}$. (You may use information from your card).

Solution. We know that $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ on $(-1, 1]$ and that integrating infinite series does not change the radius of convergence. So

$$\begin{aligned} \int_0^{1/2} \arctan x \, dx &= \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)} \Big|_0^{1/2} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{2n+2}}{(2n+1)(2n+2)} - \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{2n+2}}{(2n+1)(2n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)2^{2n+1}}. \end{aligned}$$

This series clearly alternates, while $\frac{1}{(2n+1)(2n+2)2^{2n+1}}$ is decreasing and limits to zero exactly because the denominator is an increasing function of n which grows without bound. Hence, by the Alternating Series Estimation Theorem, the error in the estimate

$$\begin{aligned} \int_0^{1/2} \arctan x \, dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)2^{2n+1}} \\ &\approx \sum_{n=0}^3 \frac{(-1)^n}{(2n+1)(2n+2)2^{2n+1}} = \frac{1}{(1)(2)2^2} - \frac{1}{(3)(4)2^4} + \frac{1}{(5)(6)2^6} - \frac{1}{(7)(8)2^8} \end{aligned}$$

is at most

$$\frac{1}{(2(4)+1)(2(4)+2)2^{2(4)+2}} = \frac{1}{(9)(10)2^{10}},$$

as required.

□

(b-10pts) Estimate $e^{1/3}$ with error $\leq \frac{3}{5! \cdot 3^5}$. (You may use information from your card).

Solution. We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

on \mathbb{R} and that the error

$$R_k(x) = e^x - \sum_{n=0}^k \frac{x^n}{n!}$$

from using the degree k Taylor polynomial to estimate the function has

$$|R_k(x)| \leq \frac{M|x|^{k+1}}{(k+1)!}$$

on $|x| < R$ if $|f^{(k+1)}(x)| \leq M$ on $|x| < R$. Of course, $f^{(k+1)}(x) = e^x$ (since e^x is so fabulous), and e^x is increasing. Thus $|e^x| < e < 3$ on $|x| < 1$ and hence

$$|R_k(x)| \leq \frac{3|x|^{k+1}}{(k+1)!}$$

on $|x| < 1$. In particular,

$$|R_4(1/3)| \leq \frac{3(1/3)^5}{5!} = \frac{3}{5! \cdot 3^5},$$

so we conclude that the estimate

$$e^{1/3} \approx \sum_{n=0}^4 \frac{(1/3)^n}{n!} = 1 + \frac{1}{3} + \frac{1}{3^2 \cdot 2!} + \frac{1}{3^3 \cdot 3!} + \frac{1}{3^4 \cdot 4!}$$

has error at most $\frac{3}{3^5 \cdot 5!}$ as required.

□