Calculus III

Math 143 Winter 2009

Professor Ben Richert

Exam 1

Solutions

Problem 1. *(20pts) Consider the sequence* $\left\{\frac{3}{\sqrt{n}}\right\}_{n=1}^{\infty}$.

(a–5pts) Does this sequence converge or diverge?

Solution. This sequence converges to zero. This is because the denominator of $\frac{3}{\sqrt{n}}$ grows arbitrarily large while the numerator is constant. $\square$

(b–5pts) What are the first 3 terms of the sequence of partial sums associated to this sequence? (Your answer to this question requires no English or simplification).

Solution. Here we have $\{S_n\} = \{\frac{3}{\sqrt{1}}, \frac{3}{\sqrt{1}} + \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{1}} + \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{3}}, \ldots\}$. $\square$

(c–10pts) Does the sequence of partial sums converge or diverge?

Solution. By definition, the sequence of partial sums diverges if and only if $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ diverges. Of course, $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ diverges by the P-test (or more particularly, $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by the P-test, and hence $\sum_{n=1}^{\infty} \frac{3}{n^{1/2}}$ diverges using the limit laws). We conclude that the sequence of partial sums diverges. $\square$

Problem 2. *(10pts) Do one of the following two problems.*

(a–10pts) Decide if $\sum_{n=1}^{\infty} (-1)^n \frac{1}{e^{n^2}}$ *converges or diverges.*

Solution. Note that this is an alternating series. So by the alternating series test, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{e^{n^2}}$ converges if the sequence $\left\{\frac{1}{e^{n^2}}\right\}$ is decreasing and limits to zero. Now $\frac{d}{dx} e^{-x^2} = -2xe^{x^2} < 0$ for all $x > 0$, and since the sequence $\frac{1}{e^{n^2}}$ lives on the function $f(x) = \frac{1}{e^{x^2}}$ it follows that $\frac{1}{e^{n^2}}$ is decreasing. Moreover, $\lim \frac{1}{e^{n^2}} = 0$ since the denominator grows without bound while the numerator is constant (we know that $\lim_{x\to\infty} e^{x^2} = \infty$). Thus, by the alternating series test, we conclude that $\sum_{n=1}^{\infty} (-1)^n \frac{1}{e^{n^2}}$ converges. $\square$

(b–10pts) Decide if $\displaystyle\sum_{n=1}^{\infty} \frac{n6^{n+1}}{n!5^n}$ *converges or diverges.*

Solution. We use the ratio test. Here

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)6^{n+1+1}}{(n+1)!5^{n+1}}}{\frac{n6^{n+1}}{n!5^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)6^{n+2}}{(n+1)!5^{n+1}}\frac{n!5^n}{n6^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)6}{(n+1)n!5}\frac{n!}{n}\right| = \lim_{n\to\infty}\left|\frac{6}{5n}\right| = 0,$$

and since $0 < 1$, we conclude that the series converges by the ratio test.

□

Problem 3. *(10pts) Do one of the following two problems.*

(a–10pts) Decide if $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-n+2}{n^3+n+1}$ *converges or diverges.*

Solution. We use the limit comparison test. Note that

$$\frac{n^2-n+2}{n^3+n+1} \geq 0.$$

Other wise there would be an n such that $n^2-n+2 < 0$. We can tell this doesn't happen by considering the function $f(x) = x^2 - x + 2$. In this case, f is continuous, positive at $x = 1$ (that is $f(1) = 2 > 0$), and increasing for $x \geq 1$ (since $f' = 2x - 1 > 0$ for $x \geq 1$), which taken together implies that $f(x) > 0$ for all $x \geq 1$. Since $n^2 - n + 2$ lives on $f(x)$ we conclude that $n^2 - n + 2$ is positive for all $n \geq 1$.

We compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$. Here

$$\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{n^2-n+2}{n^3+n+1}} = \lim_{n\to\infty}\frac{1}{n}\frac{n^3+n+1}{n^2-n+2} = \lim_{n\to\infty}\frac{n^3+n+1}{n^3-n^2+2n}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\frac{n^3+n+1}{n^3-n^2+2n} = \lim_{n\to\infty}\frac{\frac{n^3}{n^3}+\frac{n}{n^3}+\frac{1}{n^3}}{\frac{n^3}{n^3}-\frac{n^2}{n^3}+\frac{2n}{n^3}} = \lim_{n\to\infty}\frac{1+\frac{1}{n^2}+\frac{1}{n^3}}{1-\frac{1}{n}+\frac{2}{n^2}} = 1 > 0,$$

so that both series converge or both diverge by the limit comparison test. Since $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n}$ diverges by the P-test, we conclude that $\displaystyle\sum_{n=1}^{\infty}\frac{n^2-n+2}{n^3+n+1}$ diverges.

□

(b–10pts) Decide if $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^3}$ *converges or diverges.*

Solution. Note that

$$0 \leq \frac{|\sin|}{n^3} \leq \frac{1}{n^3}$$

for all $n \geq 1$ since $|\sin n|$ is always bounded by 1, and that by the P-test, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges. It follows by the comparison test that

$$\sum_{n=1}^{\infty}\frac{|\sin n|}{n^3} = \sum_{n=1}^{\infty}\left|\frac{\sin n}{n^3}\right|$$

converges, and hence by the test for absolute convergence that

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^3}$$

converges. $\square$

Problem 4. *(10pts) Find a power series representation for the function* $f(x) = \dfrac{1}{4-x^2}$ *and find its radius of convergence.*

Solution. Recall that $\sum_{n=0}^{\infty} x^n$ converges to $\dfrac{1}{1-x}$ for $|x| < 1$ (and diverges otherwise). So

$$\frac{1}{4-x^2} = \frac{1}{4}\left(\frac{1}{1-(x^2/4)}\right)$$

can be represented by

$$\frac{1}{4}\sum_{n=0}^{\infty}(x^2/4)^n = \sum_{n=0}^{\infty}\frac{x^{2n}}{4^{n+1}}$$

with radius of convergence 2 (because the series converges if and only if $|x^2/4| < 1$, i.e., if and only if $|x^2| < 4$, that is, if and only if $|x| < 2$). Here we used the fact that we can substitute $x^2/4$ for x and multiply through by the constant $1/4$ and that the latter operation does not affect the radius of convergence. $\square$

Problem 5. *(10pts) Write the Taylor Series for* $f(x) = e^x$ *centered at* $x = 1$. *(You may not use the fact that you know a power series for* e^x*).*

Solution. The formula for the Taylor series of $f(x)$ centered at $x = a$ is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In our case, $f^{(n)}(x) = e^x$ (have I mentioned yet today how fabulous e is?), so $f^{(n)}(1) = e^1 = e$, and we have that the Taylor series for $f(x) = e^x$ centered at $x = 1$ is

$$\sum_{n=0}^{\infty}\frac{e}{n!}(x-1)^n.$$

$\square$

Problem 6. *(15pts) Let* $f(x) = \cos x$.

(a–5pts) Use the Taylor Polynomial of degree 4 to estimate $f(1/10)$ *(you may use your card and you need not simplify).*

Solution. The Taylor Series for $\cos x$ (as given in the book) is,

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

and thus the degree four Taylor polynomial is

$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Then

$$\cos(1/10) \approx T_4(1/10) = 1 - \frac{(1/10)^2}{2} + \frac{(1/10)^4}{24}.$$

$\square$

(b–10pts) Is your estimate within $1/120$ *of* $\cos(1/10)$*?*

Solution. For error $= R_n(x) = f(x) - T_n(x)$, we know that

$$|R_n(x)| \le \frac{M|x|^{n+1}}{(n+1)!}$$

on $|x| < R$ where M is such that $|f^{(n+1)}(x)| \le M$ on $|x| < R$. Of course,

$$f^{(n+1)}(x) = \pm\sin x \text{ or } \pm\cos x,$$

so $|f^{(n+1)}(x)| \leq 1$ on $|x| < \infty$, and we may take $M = 1$. Thus

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

on $|x| < \infty$, or, more to the point,

$$|R_n(x)| \leq \frac{1}{(n+1)!}$$

on $|x| < 1$. For $n = 4$, this gives

$$|R_4(x)| \leq \frac{1}{5!} = \frac{1}{120},$$

and in particular,

$$|R_4(1/10)| \leq= \frac{1}{120},$$

so yes, our estimate is within 1/120 of cos(1/10).

We could also have said that since $|R_n(x)| \leq \dfrac{|x|^{n+1}}{(n+1)!}$ on $|x| < \infty$, then

$$|R_4(1/10)| \leq \frac{\left(\frac{1}{10}\right)^5}{5!} = \frac{1}{12000000} < \frac{1}{120},$$

which would also be enough. □