Methods of Proof
Math 248, Winter 2009
Professor Ben Richert

Exam 2
Solutions

Problem 1 (10pts) Let $A = \{1, 2, 3, 4, 5, 6\}$.
(a–10pts) Describe the equivalence relation $R$ on $A$ associated with the partition
\[ A = \{\{1\}, \{2, 3\}, \{4, 5, 6\}\}. \]

Solution. We have a theorem which says that the equivalence relation corresponding to a partition $A$ is $R \subset A \times A$
where $(a, b) \in R$ if and only if $a$ and $b$ are in the same element of $A$. Populating $R$ according to this rule yields:
\[ R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 3), (3, 2), (4, 5), (5, 4), (4, 6), (6, 4), (5, 6), (6, 5)\}. \]

(b–10pts) Let $S$ be the relation on $A$ given by $\{(1, 3), (2, 4), (3, 6)\}$. Describe the relation $S^{-1} \circ R$. Is $S^{-1} \circ R$ a function
from $A$ to $A$?

Solution. We have that $S^{-1} = \{(3, 1), (4, 2), (6, 3)\}$ so
\[ S^{-1} \circ R = \{(3, 1), (2, 1), (4, 2), (5, 2), (6, 2), (6, 3), (4, 3), (5, 3)\} \]
(recall that $S^{-1} \circ R = \{(a, b) \in A \times A \mid \exists c \in A$ with $(a, c) \in R$ and $(c, b) \in S^{-1}\}$). This is not a function on $A$
for several reasons—for instance, 1 never occurs as the first element of an ordered pair in $S^{-1} \circ R$ (hence even if $S^{-1} \circ R$ were a function, it could not be a function on $A$); and again, the pairs $(6, 2), (6, 3)$ are both in $S^{-1} \circ R$, but
$2 \neq 3$ (breaking rule that if $S^{-1} \circ R$ were a function then each element of $A$ occur in the first position of exactly
one ordered pair).

Problem 2 (10pts) Let $A = \{-3, -2, -1, 0, 1, 2, 3\}$ and $B = \{0, 1, 4, 9\}$. If $f : A \to B$ by $f(a) = a^2$, and $g = f|_{\{-1,3\}}$, compute $g^{-1}(\{1,9\})$.

Solution. We have that $g^{-1}(\{1,9\}) = (f|_{\{-1,3\}})^{-1}(\{1,9\}) = \{x \in \{-1,3\} \mid g(x) = f(x) = x^2 \in \{1,9\}\}$. By inspection, this
is $\{-1, 3\}$.

Problem 3 (10pts) Prove that $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ for all $n \in \mathbb{N}$.

Solution. Let
\[ S = \left\{ n \in \mathbb{N} \mid \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \right\}. \]
It is clear that $1 \in S$ since
\[ \frac{1}{(1+1)!} = \frac{1}{2} = 1 - \frac{1}{(1+1)!}. \]
Now suppose that $n \in S$. Thus
\[ \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \]
and hence
\[
\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{(n+2)!}{(n+1)!(n+2)!} + \frac{n+1}{(n+2)!} = 1 + \frac{1}{(n+2)!} = 1 - \frac{1}{(n+2)!}.
\]
Thus \(n+1 \in S\). By the principle of mathematical induction, we conclude that \(S = \mathbb{N}\), that is, that
\[
\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}
\]
for all \(n \in \mathbb{N}\).

\[\square\]

**Problem 4** (10pts) Suppose \(f : A \to B\) and \(g : B \to C\) are injective. Prove that \(g \circ f : A \to C\) is injective.

**Solution.** Recall that a function \(h\) is an injection if whenever \(x, y \in \text{Dom}(h)\) and \(h(x) = h(y)\), then \(x = y\). Now suppose that \(a, b \in A\) and \((g \circ f)(a) = (g \circ f)(b)\), that is, that \(g(f(a)) = g(f(b))\). Since \(g\) is an injection, we have \(f(a) = f(b)\). Similarly, since \(f\) is an injection, we have \(a = b\). Thus \(g \circ f\) is an injection. \(\square\)

**Problem 5** (10pts) Let \(f : A \to B\), and \(C, D \subseteq B\). Prove that \(f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D)\).

**Solution.** Recall that \(f^{-1}(C) = \{x \in A \mid f(x) \in C\}\) and likewise for \(D\) and \(C \cap D\). Now let \(x \in f^{-1}(C) \cap f^{-1}(D)\) be arbitrary. Then \(x \in f^{-1}(C), f^{-1}(D)\), and hence by definition, \(f(x) \in C\) and \(f(x) \in D\). Thus \(f(x) \in C \cap D\), and again by definition, \(x \in f^{-1}(C \cap D)\). We conclude that \(f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D)\) as required. \(\square\)

**Problem 6** (10pts) Suppose that \(R\) is a symmetric, transitive relation on a set \(A\). If the domain of \(R\) is \(A\), prove that \(R\) is an equivalence relation.

**Solution.** We need to show that for each \(a \in A\), the ordered pair \((a, a) \in R\) (i.e., we need to show that \(R\) is reflexive). So let \(a \in A\) be arbitrary. Since \(\text{Dom}(R) = A\), there is \(b \in A\) such that \((a, b) \in R\) (recall that \(\text{Dom}(R) = \{x \in A \mid \exists y \in A \text{ with } (x, y) \in R\}\)), and since \(R\) is symmetric, \((b, a) \in R\). But \(R\) is also transitive, so \((a, b), (b, a) \in R\) implies that \((a, a) \in R\).

This completes the proof. \(\square\)