Problem 1. (10pts) Suppose that $V$ is a finite dimensional vector space over $\mathbb{F}$, and $T \in \mathcal{L}(V)$ has eigenvalue 2. Show that 7 is an eigenvalue of the element $T^2 + T + I_V \in \mathcal{L}(V)$.

Solution. We must find $\overrightarrow{u} \in V - \{ \overrightarrow{0} \}$ such that $(T^2 + T + I_V)(\overrightarrow{u}) = 7 \overrightarrow{u}$. Let $\overrightarrow{v} \in V$ be an eigenvector corresponding to 2. Note that $\overrightarrow{v} \neq \overrightarrow{0}$ and $T(\overrightarrow{v}) = 2\overrightarrow{v}$. It follows that

$$(T^2 + T + I_V)(\overrightarrow{v}) = T(T(\overrightarrow{v})) + T(\overrightarrow{v}) + I_V(\overrightarrow{v}) = T(2\overrightarrow{v}) + 2\overrightarrow{v} + \overrightarrow{v} = 2T(\overrightarrow{v}) + 3\overrightarrow{v} = 2(2\overrightarrow{v}) + 3\overrightarrow{v} = 7\overrightarrow{v},$$

and we are finished. □

Problem 2. (10pts) A certain linear transformation $T \in \mathcal{L}(\mathbb{P}_1(\mathbb{R}))$ is known to have eigenvalue $\lambda = 3$ corresponding to eigenvector $1 + x$, and to send $3 + x$ to $4 + 2x$. Compute $[T]_{\beta}$ if $\beta = \{1, x\}$ is the usual standard basis of $\mathbb{P}_1(\mathbb{R})$.

Solution. There are several ways to go about this. I prefer to use the theorem which says that given ordered bases $\beta, \beta'$ of $V$, then for $Q = [I_V]_{\beta'}^{\beta}$, we have $[T]_{\beta} = Q[T]_{\beta'}Q^{-1}$. Let $\beta' = \{1 + x, 3 + x\}$, which is clearly an ordered basis. Then

$$Q = [I_V]_{\beta'}^{\beta} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

so that

$$Q^{-1} = \begin{bmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{bmatrix}.$$ 

Furthermore, since $T(1 + x)) = 3(1 + x)$ and $T(3 + x) = 4 + 2x = (1 + x) + (3 + x)$, we have

$$[T]_{\beta'} = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

and thus

$$[T]_{\beta} = Q[T]_{\beta'}Q^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} -3/2 + 2 & 9/2 - 2 \\ -3/2 + 1 & 9/2 - 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 5/2 \\ -1/2 & 7/2 \end{bmatrix}.$$

Another solution is to note that the system

$$T(1 + x) = 3 + 3x,$$

$$T(3 + x) = 4 + 2x$$

gives, by linearity

$$T(1 + x) + T(x) = 3 + 3x,$$

$$3T(1) + T(x) = 4 + 2x.$$ 

Subtracting the first equation from the second yields $2T(1) = 1 - x$, that is, $T(1) = 1/2 - x/2$, whence $T(x) = 3 + 3x - T(1) = 3 + 3x - 1/2 + x/2 = 5/2 + 7x/2$. Thus we find

$$[T]_{\beta} = \begin{bmatrix} 1/2 & 5/2 \\ -1/2 & 7/2 \end{bmatrix}.$$ 

A final solution is to note that we are trying to find $a, b, c, d \in \mathbb{R}$ such that

$$[T]_{\beta} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

We know that \([T]_{\beta}[1+x]_{\beta} = [3+3x]_{\beta}\) and \([T]_{\beta}[3+x]_{\beta} = [4+2x]_{\beta}\), that is, we know that
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}
\]
and
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}
\]
giving the system of equations
\[
\begin{align*}
a + b &= 3 \\
c + d &= 3 \\
3a + b &= 4 \\
3c + d &= 2
\end{align*}
\]
The corresponding augmented matrix is:
\[
\begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 3 \\ 3 & 1 & 0 & 0 & 4 \\ 0 & 0 & 3 & 1 & 2 \end{bmatrix}
\]
from which we read
\[
[T]_{\beta} = \begin{bmatrix} 1/2 & 5/2 \\ -1/2 & 7/2 \end{bmatrix}.
\]

\[\square\]

**Problem 3.** (15pts) Let \(V\) be a finite dimension vector space over \(F\). If \(Z\) is a subset of \(V^*\), we write
\[
Z^{-o} = \{ \overrightarrow{v} \in V \mid f(\overrightarrow{v}) = 0 \text{ for all } f \in Z \}.
\]
It turns out that \(Z^{-o}\) is a subspace of \(V\). Prove that if \(W\) is a subspace of \(V\), then \((W^o)^{-o} = W\). (Hint: for one direction, use the fact from the homework that if \(W\) is a subspace of \(V\) and \(\overrightarrow{v} \notin W\), then there is \(f \in W^o\) such that \(f(\overrightarrow{v}) \neq 0\)).

**Solution.** We first show \(W \subseteq (W^o)^{-o}\). Let \(\overrightarrow{w} \in W\). Since
\[
W^o = \{ f \in V^* \mid f(\overrightarrow{v}) = 0 \text{ for all } \overrightarrow{v} \in W \}
\]
it is immediate that \(f(\overrightarrow{w}) = 0\) for all \(f \in W^o\), that is \(\overrightarrow{w} \in \{ \overrightarrow{v} \in V \mid f(\overrightarrow{v}) = 0 \text{ for all } f \in W^o \} = (W^o)^{-o}\). We conclude that \(W \subseteq (W^o)^{-o}\) as required.

We now demonstrate that \((W^o)^{-o} \subseteq W\). Suppose \(\overrightarrow{v} \in (W^o)^{-o}\). If \(\overrightarrow{v} \notin W\), then by the homework, there is \(f \in W^o\) such that \(f(\overrightarrow{v}) \neq 0\). But for \(\overrightarrow{v} \in (W^o)^{-o}\) no such \(f\) can exist since (by definition) \(\overrightarrow{v} \in (W^o)^{-o}\) exactly because \(f(\overrightarrow{v}) = 0\) for all \(f \in W^o\). We conclude that \(\overrightarrow{v} \in W\), that is, that \((W^o)^{-o} \subseteq W\) and hence \((W^o)^{-o} = W\) as required.

\[\square\]

**Problem 4.** (15pts) Let \(V\) be a finite dimension vector space over \(F\) and \(T \in L(V)\). Prove that \(\lambda \in F\) is an eigenvalue of \(T\) if and only if \(\lambda\) is an eigenvalue of \(T^{**}\).

**Solution.** Recall from the homework that the diagram
\[
\begin{array}{ccc}
V & \xrightarrow{T} & V \\
\downarrow{\psi} & & \downarrow{\psi} \\
V^{**} & \xrightarrow{T^{**}} & V^{**}
\end{array}
\]
commutes. This means that \(T^{**} \circ \psi = \psi \circ T\) and thus if \(\overrightarrow{v} \in V\) is an eigenvector corresponding to \(\lambda\) then
\[
T^{**}(\hat{\overrightarrow{v}}) = T^{**}(\psi(\overrightarrow{v})) = \psi(T(\overrightarrow{v})) = \psi(\lambda \overrightarrow{v}) = \lambda \psi(\overrightarrow{v}) = \lambda \hat{\overrightarrow{v}}
\]
making it immediate that \(\lambda\) is an eigenvalue of \(T^{**}\) with corresponding eigenvector \(\hat{\overrightarrow{v}}\) (we know that \(\hat{\overrightarrow{v}}\) is nonzero since \(\psi\) is an isomorphism).

If you don’t want to use the commutative diagram to finish this problem, you can also proceed directly. Let \(\overrightarrow{v} \in V\) be an eigenvector corresponding to \(\lambda\) and recall that by definition \(\hat{\overrightarrow{v}}(g) = g(\overrightarrow{v})\), \(T^*(g) = g \circ T\), and hence \(T^{**}(f) = f \circ T^*\). Since
\[ \hat{v} \text{ is not zero (taking hats is an isomorphism), it is enough to show that } T^{**}(\hat{v}) = \lambda \hat{v}. \] Of course, \( T^{**}(\hat{v}) \) and \( \lambda \hat{v} \) are two functions from \( V^* \rightarrow F \), so we must demonstrate that
\[ (T^{**}(\hat{v}))(g) = \lambda \hat{v}(g) \]
for arbitrary \( g \in V^* \). We chase the definitions:
\[ (T^{**}(\hat{v}))(g) = (\hat{v} \circ T^*)(g) = \hat{v}(T^*(g)) = \hat{v}(g \circ T)(\hat{v}) = g(T(\hat{v})) = g(\lambda \hat{v}) = \lambda g(\hat{v}) = \lambda \hat{v}(g), \]
which completes the proof.

\[ \square \]

**Problem 5.** (15pts) Suppose that \( V \) is a finite dimensional vector space, \( T \in \mathcal{L}(V) \) and \( \lambda_1, \lambda_2 \in F \) are distinct eigenvalues of \( T \). If \( \vec{v}_1, \vec{v}_2 \in V \) are eigenvectors corresponding to \( \lambda_1, \lambda_2 \) respectively, prove that \( \{ \vec{v}_1, \vec{v}_2 \} \) is linearly independent. (Of course, we have a theorem which asserts this—you should prove it directly).

**Solution.** Suppose that \( \vec{v}_1 \) and \( \vec{v}_2 \) are dependent. Since both are nonzero by definition, it follows that there is \( a \in F - \{0\} \) such that \( \vec{v}_1 = a \vec{v}_2 \). But then
\[ a \lambda_1 \vec{v}_2 = \lambda(a \vec{v}_2) = \lambda_1 \vec{v}_1 = T(\vec{v}_1) = T(a \vec{v}_2) = aT(\vec{v}_2) = a \lambda_2 \vec{v}_2. \]
Since \( a \neq 0 \), we get \( \lambda_1 \vec{v}_2 = \lambda_2 \vec{v}_2 \). Now if \( \lambda_1 = 0 \), then \( \lambda_2 \vec{v}_2 = \lambda_1 \vec{v}_1 = 0 \), and since \( \lambda_2 \neq \lambda_1 = 0 \), we would have \( \vec{v}_2 = 0 \), a contradiction. Thus \( \vec{v}_2 = \frac{\lambda_2}{\lambda_1} \vec{v}_2 \). But now \( \vec{v}_2 \neq 0 \) forces \( \frac{\lambda_2}{\lambda_1} = 1 \), that is, it must be that \( \lambda_1 = \lambda_2 \), and this is a contradiction. We conclude that \( \vec{v}_1 \) and \( \vec{v}_2 \) are linearly independent as required. \[ \square \]