Problem 1. (20pts) Definitions: state precisely each of the following. You should use the definition, and not an equivalent statement.

(a—10pts) If \( f : [a, b] \to \mathbb{R} \), and \( \alpha : [a, b] \to \mathbb{R} \) is monotone increasing, state precisely what it means for \( f \) to be Riemann-Stieltjes integrable with respect to \( \alpha \) on \([a, b]\).

Solution. Given a partition \( P = \{p_0, \ldots, p_k\} \) of \([a, b]\), let
\[
M_j = \sup \{f(x) \mid x \in [p_{j-1}, p_j]\},
\]
\[
m_j = \inf \{f(x) \mid x \in [p_{j-1}, p_j]\},
\]
and
\[
\Delta \alpha_j = (\alpha(p_j) - \alpha(p_{j-1}))
\]
for all \( j = 1, \ldots, k \), and let
\[
U(f, P, \alpha) = \sum_{j=1}^{k} M_j \Delta \alpha_j
\]
and
\[
L(f, P, \alpha) = \sum_{j=1}^{k} m_j \Delta \alpha_j.
\]
Denote the infimum of the set
\[
\{U(f, P, \alpha) \mid P \text{ is a partition of } [a, b]\}
\]
by \( I^*(f) \) and the supremum of the set
\[
\{L(f, P, \alpha) \mid P \text{ is a partition of } [a, b]\}
\]
by \( I_*(f) \). We say that \( f \) is Riemann-Stieltjes integrable with respect to \( \alpha \) on \([a, b]\) if \( I^*(f) = I_*(f) \).

(b—10pts) If \( \alpha : [a, b] \to \mathbb{R} \), state precisely what it means for \( \alpha \) to be of bounded variation.
Solution. For $x \in [a, b]$, let

$$V\alpha(x) = \sup \left\{ \sum_{j=1}^{k} |\alpha(p_j) - \alpha(p_{j-1})| \left| P = \{p_0, \ldots, p_k\} \text{ is a partition of } [a, x] \right. \right\}.$$ 

We say that $\alpha$ is of bounded variation if $V\alpha(b) < \infty$. □

Problem 2. (30pts) Examples: answer each of the following with one or two sentences. Be sure to state any theorems or facts you use.

(a—10pts) Give an example of a monotone function $f : [0, 1] \to \mathbb{R}$ which is not Riemann integrable, or state why no such example exists.

Solution. We know (by a theorem in the book) that if $f : [a, b] \to \mathbb{R}$ is monotone, and $\alpha : [a, b] \to \mathbb{R}$ is monotone increasing and continuous, then the Riemann-Stieltjes integral with respect to $\alpha$ on $[a, b]$ exists; we also know that $f$ is Riemann integrable on $[a, b]$ if and only if $f$ is Riemann-Stieltjes integrable with respect to $\alpha = x$ on $[a, b]$. Since $x$ is continuous and monotone increasing on $[0, 1]$, we conclude that any monotone function $f$ must be Riemann integrable on $[0, 1]$ (since it is Riemann-Stieltjes integrable with respect to $x$ there), and hence that no such example exists. □

(b—10pts) Give an example of a function $f : [0, 1] \to \mathbb{R}_{\geq 0}$ which is Riemann integrable, but such that $\sqrt{f}$ is not Riemann integrable, or state why no such example exists.

Solution. If $f$ is Riemann integrable and positive, then it is bounded and hence has image contained in some compact interval $I \subset \mathbb{R}_{\geq 0}$; note also that $g(x) = \sqrt{x}$ is continuous on $I$, and $g(f(x)) = \sqrt{f}$. We conclude that there does not exist a function $f$ with the requested properties because we have a theorem stating that $f$ integrable on $[a, b]$ and $g(x)$ continuous on a compact interval containing the range of $f$ implies that $g(f(x))$ is integrable on $[a, b]$. □

(c—10pts) Suppose that $f(x) = \begin{cases} 0 & 0 \leq x < 1/2 \\ 1 & 1/2 \leq x \leq 1 \end{cases}$, and $\alpha = x^2$. Give an example of a partition $P$ of $[0, 1]$ such that $U(f, P, \alpha) - L(f, P, \alpha) < 1/4$, or state why no such example exists.
Solution. Consider the partition $P = \{0, \frac{2}{5}, \frac{3}{5}, 1\}$. We have that

\[
M_1 = \sup \left\{ f(x) \mid x \in \left[0, \frac{2}{5}\right] \right\} = 0
\]

\[
M_2 = \sup \left\{ f(x) \mid x \in \left[\frac{2}{5}, \frac{3}{5}\right] \right\} = 1
\]

\[
M_3 = \sup \left\{ f(x) \mid x \in \left[\frac{3}{5}, 1\right] \right\} = 1
\]

\[
m_1 = \inf \left\{ f(x) \mid x \in \left[0, \frac{2}{5}\right] \right\} = 0
\]

\[
m_1 = \inf \left\{ f(x) \mid x \in \left[\frac{2}{5}, \frac{3}{5}\right] \right\} = 0
\]

\[
m_1 = \inf \left\{ f(x) \mid x \in \left[\frac{3}{5}, 1\right] \right\} = 1
\]

and

\[
\Delta \alpha_1 = \alpha(2/5) - \alpha(0) = \left(\frac{2}{5}\right)^2 - (0)^2 = 4/25
\]

\[
\Delta \alpha_2 = \alpha(3/5) - \alpha(2/5) = \left(\frac{3}{5}\right)^2 - \left(\frac{2}{5}\right)^2 = 5/25 = 1/5
\]

\[
\Delta \alpha_2 = \alpha(1) - \alpha(3/5) = 1^2 - \left(\frac{3}{5}\right)^2 = 16/25
\]

so

\[
U(f, P, \alpha) - L(f, P, \alpha) = \sum_{j=1}^{3} M_j \Delta \alpha_j - \sum_{j=1}^{3} m_j \Delta \alpha_j
\]

\[
= \sum_{j=1}^{3} (M_j - m_j) \Delta \alpha_j = (0 - 0)(4/25) + (1 - 0)(1/5) + (1 - 1)(16/25) = 1/5 < 1/4
\]

as required.

Problem 3. (10 pts) Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous, $f(0) = 0$ and $|f'(x)| \leq x$ for all $x \in (0, 1)$. Give an upper bound for $f$ on $[0, 1]$. 

Solution. Since $f$ is continuous on $[0,1]$ and differentiable on $(0,1)$ (the latter implied by the hypothesis that $|f'(x)| \leq x$ for all $x \in (0,1)$), the mean value theorem holds. Thus, for each $b \in [0,1]$, there is $c \in (0,1)$ such that
\[
\frac{f(b)}{b} = \frac{f(b) - f(0)}{b - 0} = f'(c).
\]
We conclude that $f(b) = bf'(c) \leq b|f'(c)| \leq 1 \cdot c \leq 1$ since $b, c \leq 1$. Thus 1 is an upper bound for $f(x)$ on $[0,1]$. \qed

**Problem 4.** (15 pts) Suppose that $f : [a,b] \rightarrow \mathbb{R}$ is continuous and $\alpha : [a,b] \rightarrow \mathbb{R}$ is monotone increasing. Prove that $\int_a^b f(x) \, d\alpha$ exists.

Solution. By Riemann’s Lemma, it is enough to show that, given $\epsilon > 0$, there is a partition $P$ such that
\[
U(f, P, \alpha) - L(f, P, \alpha) < \epsilon.
\]
So let $\epsilon > 0$ be given. The function $f$ is continuous on the compact set $[a,b]$, and hence uniformly continuous. Thus there is $\delta > 0$ be such that, for $s, t \in [a,b]$, and $|s - t| < \delta$ implies $|f(s) - f(t)| < \frac{\epsilon}{\alpha(b) - \alpha(a)}$.

Let $P = \{p_0, \ldots, p_k\}$ be any partition with $m(P) < \delta$, and for each $j = 1, \ldots, k$, let
\begin{itemize}
  \item $I_j = [p_{j-1}, p_j]$,  
  \item $t_j$ be such that $f(t_j) = M_j := \sup \{f(x) \mid x \in I_j\}$, and  
  \item $s_j$ be such that $f(s_j) = m_j := \inf \{f(x) \mid x \in I_j\}$
\end{itemize}
($t_j$ and $s_j$ exist because $f$ is continuous, and hence takes its absolute max and min on a compact set). Note that
\[
|f(t_j) - f(s_j)| < \frac{\epsilon}{\alpha(b) - \alpha(a)}
\]
since $t_j, s_j \in I_j$ and the length of $I_j$ is less than $m(P)$ which is strictly less than $\delta$.

We now compute that
\[
U(f, P, \alpha) - L(f, P, \alpha) = \sum_{j=1}^k M_j \Delta \alpha_j - \sum_{j=1}^k m_j \Delta \alpha_j = \sum_{j=1}^k (M_j - m_j) \Delta \alpha_j
\]
\[
= \sum_{j=1}^k (f(t_j) - f(s_j)) \Delta \alpha_j \leq \sum_{j=1}^k |f(t_j) - f(s_j)| \Delta \alpha_j
\]
\[
< \sum_{j=1}^k \frac{\epsilon}{\alpha(b) - \alpha(a)} \Delta \alpha_j = \frac{\epsilon}{\alpha(b) - \alpha(a)} \sum_{j=1}^k \Delta \alpha_j = \frac{\epsilon}{\alpha(b) - \alpha(a)} \sum_{j=1}^k (\alpha(p_j) - \alpha(p_{j-1}))
\]
\[
= \frac{\epsilon}{\alpha(b) - \alpha(a)} \left(\alpha(p_1) - \alpha(p_0) + \alpha(p_2) - \alpha(p_1) + \cdots + \alpha(p_k) - \alpha(p_{k-1})\right)
\]
\[
= \frac{\epsilon}{\alpha(b) - \alpha(a)} (\alpha(p_k) - \alpha(p_0)) = \frac{\epsilon}{\alpha(b) - \alpha(a)} (\alpha(b) - \alpha(a)) = \epsilon,
\]
Problem 5. (15 pts) Suppose that \( f \) is Riemann integrable on \([a, b]\). Prove that \( f \) is bounded. Do not use any facts about Riemann-Stieltjes integration.

Solution. Write \( l \) to denote \( \int_a^b f(x) \, dx \), let \( \epsilon > 0 \). Then there is a partition \( P = \{p_0, \ldots, p_k\} \) such that for all choices of

\[
\{s_1, \ldots, s_k\} \in [p_0, p_1] \times \cdots \times [p_{k-1}, p_k],
\]

we have

\[
\left| \sum_{j=0}^k f(s_j)\delta_j - l \right| < \epsilon
\]

(this because \( f \) is integrable by hypothesis, and hence there is \( \delta > 0 \) such that if \( P \) is any partition with \( m(P) < \delta \), then \( |\mathcal{R}(f, P) - l| < \epsilon \)).

Now we suppose that \( f \) is not bounded, and find a contradiction. It must be that there is \( \lambda \in \{1, \ldots, k\} \) such that \( f \) is unbounded on \( I_\lambda \) (else \( f \) is certainly bounded on \([a, b]\)). For \( j = 1, \ldots, \lambda - 1, \lambda + 1, \ldots, k \), fix elements \( s_j \) in \( I_j \), and write

\[
T = \left| \sum_{j=1}^{\lambda-1} f(s_j)\Delta_j + \sum_{j=\lambda+1}^{k} f(s_j)\Delta_j \right|.
\]

For each \( s_\lambda \in I_\lambda \), we have that

\[
|f(s_\lambda)\Delta_\lambda| - |l| = |f(s_\lambda)\Delta_\lambda| - \left| \sum_{j=1}^{\lambda-1} f(s_j)\Delta_j + \sum_{j=\lambda+1}^{k} f(s_j)\Delta_j \right| - |l| \leq \left| \sum_{j=1}^{k} f(s_j)\Delta_j - l \right| < \epsilon,
\]

and it follows that for each \( s_\lambda \in I_\lambda \),

\[
|f(s_\lambda)| < \frac{\epsilon + |l| + T}{\Delta_\lambda},
\]

that is, that \( f \) is bounded on \( I_\lambda \), a contradiction. This completes the proof. \( \square \)