Calculus IV
Math 241 Winter 2006
Professor Ben Richert

Exam 2

Solutions

Problem 1. (20pts) Use the method of Lagrange multipliers to find the maximum and minimum values of \( f(x, y, z) = 2x + 4y + 6z \) given the constraint \( x^2 + y^2 + z^2 = 7/2 \).

Solution. Let \( g = x^2 + y^2 + z^2 \). According the method of Lagrange multipliers, we can find the maximum and minimal values by testing (against the function for size) each of the simultaneous solutions to the system

\[
\begin{align*}
\nabla f &= \lambda \nabla g \\
2 &= \lambda 2x \\
4 &= \lambda 2y \\
6 &= \lambda 2z \\
x^2 + y^2 + z^2 &= 7/2
\end{align*}
\]

We compute easily that \( \nabla f = (2, 4, 6) \) and \( \nabla g = (2x, 2y, 2z) \), so this system can be expanded to

\[
\begin{align*}
2 &= \lambda 2x \\
4 &= \lambda 2y \\
6 &= \lambda 2z \\
x^2 + y^2 + z^2 &= 7/2
\end{align*}
\]

Since \( \lambda \) is obviously not zero, we can rearrange the first three equations as follows

\[
\begin{align*}
1/\lambda &= x \\
2/\lambda &= y \\
3/\lambda &= z
\end{align*}
\]

and substitute the results into \( x^2 + y^2 + z^2 = 7/2 \), yielding

\[
7/2 = x^2 + y^2 + z^2 = (1/\lambda)^2 + (2/\lambda)^2 + (3/\lambda)^2 = 1/\lambda^2 + 4/\lambda^2 + 9/\lambda^2 = 14/\lambda^2.
\]

It follows that \( \lambda^2 = (2/7)14 = 4 \), and thus \( \lambda = \pm 2 \). Now if \( \lambda = 2 \), we can use the equations above to find that \( x = 1/2, y = 2/2 = 1, \) and \( z = 3/2 \), giving the critical point \((1/2, 1, 3/2)\). If \( \lambda = -2 \), we have \( x = -1/2, y = -1, \) and \( z = -3/2 \), so that \((-1/2, -1, -3/2)\) is also critical. Now test \( f \) at each of these points. We see that \( f(1/2, 1, 3/2) = 2(1/2) + 3(1) + 6(3/2) = 13 \), while \( f(-1/2, -1, -3/2) = 2(-1/2) + 3(-1) + 6(-3/2) = -13 \). We conclude that the maximum value of \( f(x, y, z) \) given the constraint \( x^2 + y^2 + z^2 = 7/2 \) is 13 (occurring at the point \((1/2, 1, 3/2)\)) while the minimum value is \(-13 \) (occurring at \((-1/2, -1, -3/2)\)).

Problem 2. (10pts) Set up an iterated integral to compute the volume of the solid inside the sphere \( x^2 + y^2 + z^2 = 1 \) and below the cone \( z = \sqrt{x^2 + y^2} \). You do not need to evaluate this integral.
Solution.

We know that if $E$ is a solid, then $\iiint_E 1 \, dV$ is the volume of $E$. In fact, this solid is not difficult to describe using spherical coordinates. First note that we want to stay inside a sphere of radius 1, so $0 \leq \rho \leq 1$. We are also required to stay underneath the cone $z = \sqrt{x^2 + y^2}$. Of course, restricting to the $xz$-plane (by setting $y = 0$) yields the equation $z = \pm x$, and hence we see that the angle between the sides of the cone and the $z$-axis is $45^\circ$ degrees, or $\pi/4$. Thus we have $\pi/4 \leq \phi \leq \pi$. Finally, we find that $0 \leq \theta \leq 2\pi$. Thus $E = \{ (\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi, \pi/4 \leq \phi \leq \pi, 0 \leq \rho \leq 1 \}$, so that the volume is

$$\int_{\pi/4}^{\pi} \int_{0}^{2\pi} \int_{0}^{1} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.$$ 

Problem 3. (10pts) Set up an iterated integral to compute the volume of the solid in the first octant bounded by $x = y^2 + z^2$ and $x = 4$. You do not need to evaluate this integral.

Solution.

We know that the volume of the solid under $f(x, y)$ and above $D$ is given by $\iint_D f(x, y) \, dA$. If we look at the solid after cutting by the plane $z = 0$ (so that $x = y^2$), we see the region $D = \{ (x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq \sqrt{x} \}$, and that the roof of the solid (if one was standing in the $xy$-plane) is $z = \sqrt{x - y^2}$. Thus the object in question is the volume under $z = \sqrt{x - y^2}$ and above $D$, and can be computed with the integral

$$\int_{0}^{4} \int_{0}^{\sqrt{x}} \sqrt{x - y^2} \, dy \, dx.$$ 

There are, of course, other ways to write this. We could have used the formula $\iiint_E 1 \, dV$, yielding $\int_{0}^{4} \int_{0}^{\sqrt{x}} \int_{0}^{\sqrt{x - y^2}} 1 \, dz \, dy \, dx$, or treated this as a type 2 region, yielding $\int_{0}^{2\pi} \int_{0}^{4} \int_{r^2}^{r^2} r \, dx \, dr \, d\theta$.

Problem 4. (10pts) Set up an iterated integral to compute the surface area of the upper face of the tetrahedron with corners $(0, 0, 0)$, $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 6)$. You do not need to evaluate this integral.

Solution.

The surface area of $f(x, y)$ above $D$ is $\int_{D} \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA$. In our situation, $f$ is the top face of the tetrahedron, which consists of the triangle through the points $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 6)$. This surface is a plane, and we know that the equation for a plane which is not parallel to the $z$-axis (as this one clearly isn’t) is $z = c - ax - by$. Furthermore, the line joining $(2, 0, 0)$ and $(0, 0, 6)$ in the $xz$-plane is on $z = c - ax - by$, and has equation $z = -3x + 6$ (use the point slope formula). Since setting $y = 0$ yields $z = c - ax$, we see that $a = 3$ and $c = 6$. Finally, because $(0, 3, 0)$ is on the plane, we know that $0 = 6 - 3(0) - b(3)$, and thus $b = 2$. We conclude that the plane in question is $z = f(x, y) = -3x - 2y + 6$. 

The base of the tetrahedron is made up of the triangle with corners \((0,0,0), (2,0,0),\) and \((0,3,0)\). The line joining \((0,3,0)\) and \((2,0,0)\) is simply \(y = -(3/2)x + 3\) (again, use the point slope formula), and hence this region is easily described as \(D = \{(x,y) \mid 0 \leq x \leq 2, \ 0 \leq y \leq -(3/2)x + 3\}\).

The partials of \(f\) are \(f_x = -3\), and \(f_y = -2\), and thus \(\sqrt{(f_x)^2 + (f_y)^2 + 1} = \sqrt{(-3)^2 + (-2)^2 + 1} = \sqrt{14}\). We can now conclude that the surface area of the top face of the tetrahedron is
\[
\int_0^2 \int_0^{-(3/2)x+3} \sqrt{14} \, dy \, dx.
\]

\[\square\]

Problem 5. (20pts) Let \(E\) be the solid in the first octant bounded by \(y = x^2, z = 1,\) and \(y = 1,\) and suppose that the density of \(E\) at \((x,y,z)\) is given by \(\rho(x,y,z) = z\) lbs per cubic inch (and, of course, \(x, y,\) and \(z\) are in inches). Find \(\tau,\) the center of mass of this solid in the \(z\)-direction.

**Solution.** We know that if the density of \(E\) at \((x,y,z)\) is \(\rho(x,y,z),\) then
\[
\text{mass} = \iiint_E \rho(x,y,z) \, dV
\]
and
\[
\tau = \frac{M_{xy}}{\text{mass}} = \frac{1}{\text{mass}} \iiint_E z \rho(x,y,z) \, dV.
\]

In our situation, \(E\) is fairly simple to describe, \(E = \{(x,y,z) \mid 0 \leq z \leq 1\}\) with \(D = \{(x,y) \mid x^2 \leq y \leq 1, \ 0 \leq x \leq 1\},\) so \(E = \{(x,y,z) \mid x^2 \leq y \leq 1, \ 0 \leq x \leq 1, \ 0 \leq z \leq 1\}.

It follows that
\[
\text{mass} = \int_0^1 \int_{x^2}^1 \int_0^1 z \, dz \, dy \, dx = \int_0^1 \int_{x^2}^1 \left(\frac{z^2}{2}\right)_0^1 \, dy \, dx
\]
\[
= \int_0^1 \int_{x^2}^1 \frac{1}{2} \, dy \, dx = \int_0^1 \left(y/2\right)_{x^2}^1 \, dx = \int_0^1 \frac{1}{2} (1-x^2/2) \, dx = (x/2 - x^3/6)|_0^1 = 1/2 - 1/6 = 1/3 \text{ lbs},
\]
and
\[
\tau = \frac{1}{3} \int_0^1 \int_{x^2}^1 \int_0^1 z^2 \, dz \, dy \, dx = 3 \int_0^1 \int_{x^2}^1 \left(\frac{z^3}{3}\right)_0^1 \, dy \, dx = 3 \int_0^1 \int_{x^2}^1 1/3 \, dy \, dx
\]
\[
= \int_0^1 \int_{x^2}^1 1 \, dy \, dx = \int_0^1 \left(y\right)_{x^2}^1 \, dx = \int_0^1 (1-x^2) \, dx = (x-x^3/3)|_0^1 = (1-1/3) = 2/3 \text{ in.}
\]

\[\square\]

Problem 6. (20pts) Suppose that \(t(x,y)\) gives the depth in feet of topsoil \(x\) feet east and \(y\) feet north of the southwest corner of a certain 100 foot by 100 foot square field out on the crops unit.

(a-10pts) Devise a Riemann sum which computes the total volume of topsoil on the field. Be sure to explain why your sum computes volume. (Hint: just like we did several times in class).

**Solution.** Split the interval \([0,100]\) in the \(x\) direction into \(n\) pieces of width \(\Delta x = \frac{100}{n}\) and label the endpoints with \(x_i = i\Delta x\). Similarly, split the interval \([0,100]\) in the \(y\) direction into \(m\) pieces of width \(\Delta y = \frac{100}{m}\) and label the endpoints with \(y_j = j\Delta y\). We note that this operation partitions the field into \(nm\) rectangles. If the depth of
the topsoil under each rectangle was constant, then we could compute the volume under \([x_{i-1}, x_i] \times [y_{j-1}, y_j]\) to be \(t(x_i, y_j) \Delta x \Delta y\) (which is the area of \([x_{i-1}, x_i] \times [y_{j-1}, y_j]\) times the depth at the northeast corner, \(t(x_i, y_j)\)). As it stands, this approximates the volume of topsoil under \([x_{i-1}, x_i] \times [y_{j-1}, y_j]\), and hence the total volume of topsoil on the field is approximately \(\sum_{i=1}^{n} \sum_{j=1}^{m} t(x_i, y_j) \Delta x \Delta y\). We notice that the smaller the rectangles, the less egregious our supposition that the topsoil has constant depth there, and therefore believe that the volume is equal to the limit: 

\[
\lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} t(x_i, y_j) \Delta x \Delta y.
\]

\(\Box\)

(b-10pts) After taking some data readings (at great risk to yourself), you find the following values for \(t(x, y)\):

<table>
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<tr>
<th>(x,y )</th>
<th>0</th>
<th>25</th>
<th>50</th>
<th>75</th>
<th>100</th>
</tr>
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<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>25</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
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<td>1</td>
<td>3</td>
<td>2</td>
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<tr>
<td>100</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Use the Midpoint rule with \(n = m = 2\) to estimate the volume of topsoil on the field.

**Solution.** We know from part (a), that the volume of topsoil on the field is 

\[
\lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} t(x_i, y_j) \Delta x \Delta y.
\]

We now estimate the amount of topsoil by taking \(n = m = 2\), and using the midpoint rule, which samples the depth in the \(ij\)th rectangle not at the northeast corner, but at the point in the center \((\overline{x}_i, \overline{y}_j) = \left(\frac{x_i - x_{i-1}}{2}, \frac{y_j - y_{j-1}}{2}\right)\). We have \(\Delta x = \Delta y = (100 - 0)/2 = 50\), \(\overline{x}_1 = \overline{y}_1 = 25\), and \(\overline{x}_2 = \overline{y}_2 = 75\), so there is approximately

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} t(\overline{x}_i, \overline{y}_j) \Delta x \Delta y = t(25, 25)50^2 + t(25, 75)50^2 + t(75, 25)50^2 + t(75, 75)50^2
\]

\[
= 2(2500) + 2(2500) + 2(2500) + 3(2500) = 22500
\]
cubic feet of topsoil on the field. \(\Box\)