Problem Set 4, due Thursday, January 29

4.42 Prove that an infinite group must have an infinite number of subgroups.

If $G$ contains only finitely many subgroups, then choose a list of elements $a_1, a_2, \ldots$ such that: $a_1 \in G$, $a_2 \in G - \langle a_1 \rangle$, $a_3 \in G - \{\langle a_1 \rangle \cup \langle a_2 \rangle \}$, and so on. Note that $\langle a_i \rangle \neq \langle a_j \rangle$ for $i \neq j$ by construction. Then because there are only finitely many subgroups it must be that for some $t$ the set $G - \{\langle a_1 \rangle \cup \ldots \cup \langle a_t \rangle \}$ is empty. Then as a set,

$$G = \bigcup_{i=1}^{t} \langle a_i \rangle.$$ 

This can only occur if one of the $\langle a_i \rangle$, call it $\langle a \rangle$, has infinite order (otherwise we have an infinite set written as the union of finitely many finite sets). So $G$ contains an element with infinite order.

Now consider the subgroups $\langle a \rangle, \langle a^2 \rangle, \ldots$. Because $G$ contains only finitely many groups, there must be some $i, j \in \mathbb{Z}^+$ such that $i \neq j$ and $\langle a^i \rangle = \langle a^j \rangle$. But this implies that $a^i = (a^j)^t$ and $a^j = (a^i)^t$. By theorem 4.1, all powers of an infinite element are distinct, so we have that $i = jt$ and $j = it$. This implies that $i = j$, a contradiction. We conclude that $G$ must have infinitely many subgroups.

4.46 Let $|x| = 40$. List all the elements of $\langle x \rangle$ that have order 10.

According to theorem 4.2, $|a^k| = n/(n, k)$ for $|a| = n$. Thus the elements in $\langle x \rangle$ of order 10 correspond to integers between $1 \leq t \leq 40$ such that $40/(40, t) = 10$. These in turn correspond to $t$ such that $(40, t) = 4$, that is, to $t = 4, 12, 28, \text{ and } 32$. So the elements in question are $x^4, x^{12}, x^{28}, \text{ and } x^{32}$.

4.51 Prove that no group can have exactly two elements of order 2.

Suppose that $a, b \in G$ are two distinct elements of order 2 in a group $G$. Note that because $a \neq b$, it cannot be the case that $ab = e$. By cancellation, $ab = a$ and $ab = b$ are also ruled out. If $ab = ba$, then $(ab)^2 = abab = aabb = e$ and $ab$ is a third element of order two. If $ab \neq ba$, than an immediate consequence is that $aba \neq b$, and we also know that $aba \neq e$ and $aba \neq a$ because $ba \neq a$ and $ba \neq e$. Thus $aba$ is distinct from $e$, $a$, and $b$. But $(aba)^2 = ababa = abba = aa = e$, so $aba$ is a third element of order 2 in $G$. We conclude that no group can have exactly two elements of order 2.