4.1.6 Which of the following subsets of $R[x]$ are subrings of $R[x]$? Justify your answer:

If $R = 0$, then all these are subrings, so let us assume that $R \neq 0$

(a) All polynomials with constant term 0$_R$.

This is a subring (without multiplicative identity) of $R[x]$. To verify this we need only to check that it contains zero, and that is closed under subtraction and multiplication. Note that zero is obviously a polynomial with constant 0$_R$. Furthermore, the product or difference of two polynomials whose constant terms are zero is again a polynomial with zero constant term, so we are done.

(b) All polynomials of degree 2.

This is not a subring because 0$_R$ is not a polynomial of degree 2 (any subring must contain zero).

(c) All polynomials of degree $\leq k$, where $k$ is a fixed positive integer.

If $R$ is such that for all $a, b \in R$, $ab = 0$, then this is a subring because $(a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0)(b_k x^k + b_{k-1} x^{k-1} + \cdots + b_0)$ always equals zero (so the subset is closed under multiplication), and the difference between two degree $\leq k$ polynomials is always degree $\leq k$.

If $R$ contain $a, b$ such that $ab \neq 0$, then this is not a subring because it is not closed under multiplication (for instance, $(ax^k)(bx^k) = abx^{2k}$ is not a polynomial of degree $\leq k$).

(d) All polynomials in which the odd powers of $x$ have zero coefficients.

This is a subring of $R$. We need to check for the zero element and for closure under multiplication and subtraction. The zero polynomial has all coefficients zero, so it is in this subset. Also, if $f$ and $g$ are polynomials of the prescribed type, then so is $f - g$ (if all the coefficients of odd powers are zero in $f$ and $g$, then the same must be true for $f - g$ because the coefficient of any given power is equal to the difference of the coefficients.
for that power from \( f \) and \( g \). To check multiplicative closure, note that multiplying \( rx^i \) and \( sx^j \) multiplies the coefficients and adds exponents. So if \( f \) and \( g \) have the prescribed properties, all odd powers in \( fg \) have zero coefficients (each results as the sum of a product of an even term from \( f \) times an odd term from \( g \) or vice-versa).

(e) All polynomials in which the even powers of \( x \) have zero coefficients.

If \( R \) is such that for all \( a, b \in R \), \( ab = 0 \), then this is a subring of \( R \). Note that zero clearly satisfies the required property, and that all products are zero, so it is closed under multiplication. Also, if \( f \) and \( g \) are polynomials of the prescribed type, then so is \( f - g \) (if all the coefficients of even powers are zero in \( f \) and \( g \), then the same must be true for \( f - g \) because the coefficient of any given power is equal to the difference of the coefficients for that power from \( f \) and \( g \)).

If \( R \) contain \( a, b \) such that \( ab \neq 0 \), then this is not a subring because it is not closed under multiplication (for instance, \((ax)(bx) = abx^2\) is not a polynomial whose even powers have zero coefficients.

4.1.16 Let \( \phi : \mathbb{Z}[x] \rightarrow \mathbb{Z}[x] \) be the function that maps the polynomial \( a_0 + a_1 x + \cdots + a_k x^k \) in \( \mathbb{Z}[x] \) onto the polynomial \([a_0] + [a_1] x + \cdots + [a_k] x^k\), where \([a]\) denotes the class of the integer \( a \) in \( \mathbb{Z}_n \). Show that \( \phi \) is a subjective homomorphism of rings.

Let \( f, g \in \mathbb{Z}[x] \), where \( f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \) and \( g = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0 \). We can suppose without loss of generality that \( n > m \). Then

\[
\phi(f + g) = \phi((a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_0 + b_0))
= [a_n + b_n]x^n + [a_{n-1} + b_{n-1}]x^{n-1} + \cdots + [a_0 + b_0]
= ([a_n] + [b_n])x^n + ([a_{n-1}] + [b_{n-1}])x^{n-1} + \cdots + ([a_0] + [b_0])
= [a_n]x^n + [a_{n-1}]x^{n-1} + \cdots + [a_0] + [b_m]x^m + [b_{m-1}]x^{m-1} + \cdots + [b_0]
= \phi(f) + \phi(g).
\]

Furthermore,

\[
\phi(fg) = \phi \left( \sum_{i=0}^{n+m} \sum_{j=0}^{i} a_j b_{i-j} x^{i} \right)
= \sum_{i=0}^{n+m} \sum_{j=0}^{i} [a_j b_{i-j}] x^{i}
\]
\[
\sum_{i=0}^{n+m} \sum_{j=0}^{i} a_j[b_{i-j}]x^i
\]

\[
= ([a_n]x^n + [a_{n-1}]x^{n-1} + \cdots + [a_0])([b_n]x^n + [b_{n-1}]x^{n-1} + \cdots + [b_0])
\]

\[
= \phi(f)\phi(g).
\]

We conclude that \(\phi\) is a homomorphism.

Now let \(h = [c_n]x^n + [c_{n-1}]x^{n-1} + \cdots + [c_0]\) be an element in \(\mathbb{Z}_2[x]\). Because each \(c_i\) is an element of \(\mathbb{Z}\), we have that \(\phi(c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0) = [c_n]x^n + [c_{n-1}]x^{n-1} + \cdots + [c_0] = h\), and conclude that \(\phi\) is onto. This completes the proof.

4.1.18 Let \(D : \mathbb{R}[x] \to \mathbb{R}[x]\) be the derivative map defined by

\[
D(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}.
\]

Is \(D\) a homomorphism of rings? An isomorphism?

Note that \(D(x^2x) = D(x^3) = 3x^2 \neq 2x = D(x^2)D(x)\), so this is not a homomorphism, and thus is not an isomorphism.

4.2.8 Let \(f(x), g(x) \in F[x]\), not both zero, and let \(d(x)\) be their gcd. If \(h(x)\) is a common divisor of \(f(x)\) and \(g(x)\) of highest possible degree, then prove that \(h(x) = cd(x)\) for some nonzero \(c \in F\).

Obviously \(\deg(h) \geq \deg(d)\). In fact, it must be the case that \(\deg(h) = \deg(d)\), because if \(\deg(h) > \deg(d)\), then \(d\) would not be the greatest common divisor. By corollary 4.5, we know that \(d(x) \mid h(x)\). So \(dg = h\), and thus because \(\deg(h) = \deg(d) + \deg(g)\), it must be that \(\deg(g) = 0\), in other words, that \(g\) is a constant. That does it.

4.2.14 Let \(f(x), g(x), h(x) \in F[x]\), with \(f(x)\) and \(g(x)\) relatively prime. If \(f(x) \mid h(x)\) and \(g(x) \mid h(x)\), prove that \(f(x)g(x) \mid h(x)\).

Because \(f \mid h\), there is a \(t \in F[x]\) such that \(ft = h\). Then \(g \mid ft\). By theorem 4.7, we must have that \(g \mid t\). So there is \(k \in F[x]\) such that \(gk = t\). Finally, \(fgk = ft = h\), so \(fg \mid h\) as required.
4.2.16 Let \( f(x), g(x), h(x) \in F[x] \), with \( f(x) \) and \( g(x) \) relatively prime. Prove that the gcd of \( f(x)h(x) \) and \( g(x) \) is the same as the gcd of \( h(x) \) and \( g(x) \).

Obviously \((h, g)\) divides \((fh, g)\), so it is enough to show that \((fh, g) \mid (h, g)\). If not, then there is a \( t \in F[x] \), \( t \) not a constant, such that \( t \mid g \), \( t \mid fh \), and \( t \nmid h \). We conclude that \((t, f) \neq 1\) (else \( t \mid h \) by theorem 4.7). Let \( k = (t, f) \) (note that \( k \) is not a constant). Then \( k \mid t \mid g \) and \( k \mid f \), so \((f, g) \neq 1\), a contradiction. We conclude that \((fh, g) \mid (h, g)\) as required.