1. Recall the definitions of the direct image \( f[X] \) and inverse image \( f^{-1}[Y] \) from class: If \( f : A \to B \), then \( f[X] = \{ f(x) \in B \mid x \in X \} \), and \( f^{-1}[Y] = \{ x \in A \mid f(x) \in Y \} \). Let \( X, X' \subset A \), and \( Y \subset B \). Prove the following:

(a) \( f[X \cup X'] = f[X] \cup f[X'] \) (new and improved)
Let \( y \) be an element in \( f[X \cup X'] \). Then by definition there exists an element \( x \in (X \cup X') \) such that \( f(x) = y \). Furthermore, \( x \in X \) or \( x \in X' \). Without loss of generality we may assume that \( x \in X \). Thus \( y = f(x) \in f[X] \cup f[X'] \). We conclude that \( f[X \cup X'] \subset f[X] \cup f[X'] \).

Suppose on the other hand that \( y \in f[X] \cup f[X'] \). Thus \( y \in f[X] \) or \( y \in f[X'] \). We may assume without loss of generality that \( y \in f[X] \). Then there exists an \( x \in X \) such that \( f(x) = y \), and \( y = f(x) \in f[X] \subset f[X] \cup f[X'] \). We conclude that \( f[X] \cup f[X'] \subset f[X] \cup f[X'] \), and therefore that \( f[X \cup X'] = f[X] \cup f[X'] \).

(b) \( X \subset f^{-1}[f[X]] \)
Let \( x \in X \). Then it is obvious that \( x \in f^{-1}[f[X]] = \{ x \in A \mid f(x) \in f[X] \} \) because our element \( x \) is an element of \( A \) such that \( f(x) \in f[X] \).

To show that equality does not hold in general, we must show that there exists sets \( A \), \( B \), and \( X \) and a function \( f \) such that \( X \subset A \), but \( f^{-1}[f[X]] \not\subset X \). Let \( A = \{ a, b \} \), and \( B = \{ c \} \) be sets with two and one elements respectively. Suppose that \( X = \{ a \} \). If \( f \) is the function from \( A \) to \( B \) such that \( f(a) = c = f(b) \), then \( f^{-1}[f[X]] = f^{-1}[f(\{ a \})] = f^{-1}[\{ c \}] = \{ a, b \} \not\subset \{ a \} = X \). Thus we know that in general, \( X \neq f^{-1}[f[X]] \).

(c) \( f[f^{-1}[Y]] \subset Y \)
Let \( y \) be an element in \( f[f^{-1}[Y]] = \{ f(x) \in B \mid x \in f^{-1}[Y] \} \). This means that there exists an \( x \in f^{-1}[Y] \) such that \( f(x) = y \). But \( f^{-1}[Y] = \{ x \in A \mid f(x) \in Y \} \) so that \( x \in f^{-1}[Y] \) implies that \( y = f(x) \in Y \). We conclude that \( f[f^{-1}[Y]] \subset Y \).

To show that equality does not hold in general, we must show that there exists sets \( A \), \( B \), and \( Y \), and a function \( f \) such that \( Y \subset B \), but \( Y \not\subset f[f^{-1}[X]] \). Let \( A = \{ a \} \) and \( B = \{ b, c \} \) be sets with one and two elements respectively. Suppose that \( Y = \{ c \} \). If \( f \) is the function from \( A \) to \( B \) such that \( f(a) = b \), then \( f[f^{-1}[Y]] = f[f^{-1}[\{ c \}]] = f[\{ c \}] = f[\phi] = \phi \), and this clearly fails to contains \( Y \). Thus we know that in general \( f[f^{-1}[Y]] \neq Y \).

In parts (b) and (c) give a counterexample to show that equality does not hold in general.

2. Prove that for all \( n \geq 4 \), \( n! \geq 2^n \).
Let \( P(n) \) be the statement that \( n! \geq 2^n \) for all \( n \geq 4 \). This statement is clearly true for \( n = 4 \), because \( 4! = 24 \geq 16 = 2^4 \). Suppose then that \( P(n) \) is true for some \( n \geq 4 \). So \( n! \geq 2^n \) and multiplying both side by \( (n + 1) \) we find that \( (n + 1)n! \geq 2^n(n + 1) \). But \( 2^n(n + 1) > 2^{n+1} \) because \( n \geq 4 \). Thus \( (n + 1)! \geq 2^{n+1} \). We conclude that, if \( P(n) \) is true, then \( P(n + 1) \) must also be true. By Induction (1) this implies that \( P(n) \) must be true for all \( n \geq 4 \). This completes the proof.

3. Prove that \( 1^3 + 2^3 + \cdots + n^3 = \frac{n(n+1)}{2}^2 \) for all \( n \geq 1 \).
Let \( P(n) \) be the statement that \( 1^3 + 2^3 + \cdots + n^3 = \frac{n(n+1)}{2}^2 \) for all \( n \geq 1 \). This statement is clearly true for \( n = 1 \), because \( 1^3 = 1 = [1(1+1)/2]^2 \). Suppose then that \( P(n) \) is true for some \( n \geq 1 \). Then \( 1^3 + 2^3 + \cdots + n^3 = [n(n+1)/2]^2 \), and adding \( (n + 1)^3 \) to both sides of the equation yields the equality \( 1^3 + 2^3 + \cdots + n^3 + (n + 1)^3 = [n(n+1)/2]^2 + (n + 1)^3 \). Notice that

\[
\left( \frac{n(n+1)}{2} \right)^2 + (n+1)^3 = (n+1)^2 \left( \frac{n^2}{4} + 4 \left( \frac{n+1}{4} \right) \right) = (n+1)^2 \left( \frac{n^2 + 4n + 4}{4} \right)
\]

\[
= (n+2) \left( \frac{(x+2)^2}{4} \right) = \left( \frac{(x+1)(x+2)}{2} \right)^2.
\]
Thus \(1^3 + 2^3 + \cdots + (n+1)^3 = [(n+1)(n+2)/2]^2\) and we conclude that if \(P(n)\) is true then \(P(n+1)\) must also be true. By Induction (1) this implies that \(P(n)\) must be true for all \(n \geq 1\). This completes the proof.

4. (a) Prove that
\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^{n-1} + \frac{x^n}{1-x}
\]
for any \(n \geq 1\).
Let \(P(n)\) be the statement that
\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^{n-1} + \frac{x^n}{1-x}
\]
for any \(n \geq 1\). This statement is clearly true for \(n = 1\), because
\[
\frac{1}{1-x} = 1 + \frac{x}{1-x}.
\]
(Note that \(1 + \frac{x}{1-x} = \frac{1-x}{1-x} + \frac{x}{1-x} = \frac{1}{1-x}\).
Suppose then that \(P(n)\) is true for some \(n \geq 1\). Thus
\[
1 + x + x^2 + \cdots + x^{n-1} + x^n + \frac{x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x} + x^n + \frac{x^{n+1}}{1-x}
\]
because
\[
\frac{1}{1-x} - \frac{x^n}{1-x} = 1 + x + x^2 + \cdots + x^n
\]
by the induction hypothesis. But
\[
\frac{1}{1-x} - \frac{x^n}{1-x} + \frac{x^{n+1}}{1-x} = \frac{1-x^n + x^n - x^{n+1} + x^{n+1}}{1-x} = \frac{1}{1-x}
\]
We conclude that
\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x}
\]
so that if \(P(n)\) is true then \(P(n+1)\) must also be true. By Induction (1) this implies that \(P(n)\) must be true for all \(n \geq 1\). This completes the proof.

(b) Prove that \(1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1\) for any \(n \geq 1\).
Take \(x = 2\) in the formula from part (a). Thus
\[
\frac{1}{1-2} = 1 + 2 + 2^2 + \cdots + 2^{n-1} + \frac{2^n}{1-2},
\]
or, rearranging things,
\[-1 = 1 + 2 + 2^2 + \cdots + 2^{n-1} - 2^n,
\]
so that
\[
2^n - 1 = 1 + 2 + 2^2 + \cdots + 2^{n-1}.
\]

(c) Given any integer \(n > 1\) and any integers \(a, r_0, r_1, \ldots, r_{n-1}\) with \(a \geq 2\) and \(0 \leq r_i < a\) for all \(i = 0, 1, \ldots, n-1\), prove, using (a), that
\[
r_0 + r_1 a + r_2 a^2 + \cdots + r_{n-1} a^{n-1} < a^n.
\]
Take \(x = a\) in the formula from part (a). Then
\[
\frac{1}{1-a} = 1 + a + a^2 + \cdots + a^{n-1} + \frac{a^n}{1-a},
\]
so that
\[
\frac{a^n}{1-a} + \frac{1}{1-a} = 1 + a + a^2 + \cdots + a^{n-1}.
\]
Note that \(-\frac{a^n}{1-a} = \frac{a^n}{a-1}\). After multiplying through by \((a-1)\) we obtain
\[ a^n - 1 = (a-1) + (a-1)a + (a-1)a^2 + \cdots + (a-1)a^{n-1}. \]
But \(r_i < a\), so \((a-1)a^i \geq r_i a^i\) for \(i = 0, \ldots, n\). Thus
\[ a^n > a^n - 1 = (a-1) + (a-1)a + (a-1)a^2 + \cdots + (a-1)a^{n-1} \geq r_0 + r_1a + r_2a^2 + \cdots + r_{n-1}a^{n-1}, \]
as required.

5. Prove that if Induction(1) is true then Induction(2) is true. By Induction(1) I mean that formulation I first introduced on Friday, 1/5. By Induction(2) I mean that formulation first introduced Monday, 1/8.

Suppose that \(P(n)\) is a statement which makes sense for all \(n \geq n_0\). Suppose further that \(P(n_0)\) is true, and that whenever \(P(m)\) is true for \(n_0 \leq m \leq n, n \geq n_0\), then \(P(m+1)\) must also be true. In order to show that Induction (2) is true, I must show that these hypothesis imply that \(P(n)\) is true for all \(n \geq n_0\). To do this we use Induction (1).

Let \(Q(n)\) be the statement that \(P(m)\) is true for all \(n_0 \leq m \leq n\). It is clear that \(Q(n_0)\) is true, because \(Q(n_0)\) is the statement that \(P(n_0)\) is true. Suppose then that \(Q(n)\) is true for some \(n \geq n_0\). By the definition of \(Q(n)\), this means that \(P(m)\) is true for all \(n_0 \leq m \leq n\). By our hypothesis, then, \(P(n+1)\) is true, and thus \(Q(n+1)\) is true (because \(Q(n+1)\) is the statement that \(P(m)\) is true for all \(n_0 \leq m \leq n + 1\), which is true). We conclude that if \(Q(n)\) is true, then \(Q(n+1)\) must also be true, and thus by Induction (1) that \(Q(n)\) is true for all \(n \geq n_0\). This implies, again by the definition of \(Q(n)\), that \(P(n)\) is true for all \(n \geq n_0\). This completes the proof.

6. Prove that the sum of the elements of the \(n\)th row of Pascal’s triangle is \(2^n\) for each \(n\). How many subsets of a set with \(n\) elements are there?

Let \(P(n)\) be the statement that \(\sum_{i=0}^n C(n, i) = 2^n\), that is, that the sum of the elements in the \(n\)th row of Pascal’s triangle is \(2^n\). This statement is clearly true for \(n = 0\), because there is exactly one entry in the “zero-th” row of Pascal’s triangle, \(C(0, 0) = 1 = 2^0\). Suppose then that \(P(n)\) is true for some \(n \geq 0\). Now by the definition of Pascal’s triangle, \(\sum_{i=0}^{n+1} C(n + 1, i) = C(n + 1, 0) + \sum_{i=1}^{n+1} (C(n, i-1) + C(n, i)) + C(n + 1, n + 1)\). Then because \(C(n + 1, 0) = 1 = C(n, 0)\) and \(C(n + 1, n + 1) = 1 = C(n, n)\), we know that
\[
C(n + 1, 0) + \sum_{i=1}^{n} C(n, i-1) + \sum_{i=1}^{n} C(n, i) + C(n + 1, n + 1) = C(n, 0) + \sum_{i=1}^{n} C(n, i-1) + \sum_{i=1}^{n} C(n, i) + C(n + 1, n + 1) = 2^n + 2^n.
\]
The last equality is by the induction hypothesis. But \(2^n + 2^n = 2^{n+1}\), so we conclude that
\[
\sum_{i=0}^{n+1} C(n + 1, i) = 2^{n+1},
\]
that is, if \(P(n)\) is true then \(P(n + 1)\) must also be true. By Induction (1) this implies that \(P(n)\) must be true for all \(n \geq 1\). This completes the proof.

Recall that in class we showed that \(C(n, i) = \binom{n}{i}\) and that \(\binom{n}{i}\) counted the number of ways to choose an \(i\) element subset from a set of \(n\) elements (that second part was left as an exercise for you). It follows that the total number of subsets of a set with \(n\) elements would be the number of zero element subsets, plus the number of one element subsets, plus the number of two element subsets, and so on up to \(n\). This is precisely \(\sum_{i=0}^{n} C(n, i)\) which we have shown is \(2^n\).