1. Recall the definitions of the direct image $f[X]$ and inverse image $f^{-1}[Y]$ from class: If $f : A \rightarrow B$, then $f[X] = \{f(x) \in B \mid x \in X\}$, and $f^{-1}[Y] = \{x \in A \mid f(x) \in Y\}$. Let $X, X' \subset A$, and $Y \subset B$. Prove the following:

(a) $f[X \cup X'] = f[X] \cup f[X']$ (new and improved)
(b) $X \subset f^{-1}[f[X]]$
(c) $f[f^{-1}[Y]] \subset Y$.

In parts (b) and (c) give a counterexample to show that equality does not hold in general.

2. Prove that for all $n \geq 4$, $n! \geq 2^n$.

3. Prove that $1^3 + 2^3 + \cdots + n^3 = \frac{n(n+1)^2}{2}$ for all $n \geq 1$.

4. (a) Prove that
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^{n-1} + \frac{x^n}{1-x}$$
for any $n \geq 1$.

(b) Prove that $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$ for any $n \geq 1$.

(c) Given any integer $n > 1$ and any integers $a, r_0, r_1, \ldots, r_{n-1}$ with $a \geq 2$ and $0 \leq r_i < a$ for all $i = 0, 1, \ldots, n - 1$, prove, using (a), that
$$r_0 + r_1 a + r_2 a^2 + \cdots + r_{n-1} a^{n-1} < a^n.$$