Problem 1. (10pts) Show that \( y = \frac{\ln x + 19}{x} \) is a solution to the differential equation \( x^2 y' + xy = 1 \).

Solution. Let \( F(x, y, y') = x^2 y' + xy - 1 \). Then to show that \( y \) is a solution, we need to demonstrate that
\[
F\left(x, \frac{\ln x + 19}{x}, \frac{d}{dx} \left( \frac{\ln x + 19}{x} \right) \right) = 0.
\]
Using the quotient rule, we compute that
\[
\frac{d}{dx} \left( \frac{\ln x + 19}{x} \right) = \frac{x \left( \frac{1}{x} \right) - 1 (\ln x + 19)}{x^2} = \frac{1 - \ln x - 19}{x^2}
\]
and thus
\[
F\left(x, \frac{\ln x + 19}{x}, \frac{1 - \ln x - 19}{x^2} \right) = x^2 \left( \frac{1 - \ln x - 19}{x^2} \right) + x \left( \frac{\ln x + 19}{x} \right) - 1 = 1 - \ln x - 19 + \ln x + 19 - 1 = 0
\]
as required.
Problem 2. (8pts) Match each of the differential equations with its slope field: You are not required to show any work for this problem.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Slope Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A) ( y' = x - y + 1 )</td>
<td>III</td>
</tr>
<tr>
<td>(B) ( y' = x + y + 1 )</td>
<td>IV</td>
</tr>
<tr>
<td>(C) ( y' = \sin x + \sin y )</td>
<td>II</td>
</tr>
<tr>
<td>(D) ( y' = \frac{4 - 2x}{3y^2 - 5} )</td>
<td>I</td>
</tr>
</tbody>
</table>
Problem 3. (10pts) Solve the initial value problem:
\[
\frac{dy}{dx} = \frac{\sin x}{e^y + 1}, \quad y(\pi/2) = 0.
\]
An implicit solution is acceptable.

Solution. We begin by assuming that we can cross multiply and obtain (the lie)
\[(e^y + 1)dy = \sin x \, dx.
\]
We then assume we can integrate both sides, which yields:
\[
\int (e^y + 1) \, dy = \int \sin x \, dx.
\]
So
\[e^y + y = \int (e^y + 1) \, dy = \int \sin x \, dx = -\cos x + C
\]
and the fact that \((\pi/2, 0)\) is on the solutions implies that
\[1 = e^0 + 0 = -\cos(\pi/2) + C = C,
\]
or \(C = 1\), whence the implicit solution is \(e^y + y = -\cos x + 1\).

Problem 4. (15pts) Water with a concentration of 1 pound of salt per gallon flows into a tank at a rate of 2 gallon per minute, mixes instantaneously, and flows out at a rate of 1 gallon per minute. Initially, the tank holds 10 gallons of water which contains 1 pound of salt. Develop a function which describes the amount of salt in the tank after \(t\) minutes.

Solution. Let \(y(t)\) be the amount of salt (in pounds) in the tank at time \(t\), (whence \(y(0) = 1\)). We know that for a short time interval \(\Delta t\),
\[
\Delta y = \text{(amount of salt in)} - \text{(amount of salt out)} \\
\approx \text{(rate water flows in)(concentration of salt in the water flowing in)(time)} \\
- \text{(rate water flows out)(concentration of salt in the water flowing out)(time)} \\
\approx (2)(1)\Delta t - (1)(\text{concentration of salt in the water flowing out})\Delta t.
\]
Now the concentration of salt in the water flowing out of the tank during the time interval \(\Delta t\) is approximately
\[
\frac{y}{\text{volume of water in the tank}} = \frac{y}{10 + t}
\]
since there are initially 10 gallons of water in the tank, and each minute one more gallon flows in then flows out. So
\[
\Delta y \approx 2\Delta t - \frac{y(t)}{10 + t} \Delta t,
\]
and we conclude that
\[
\frac{\Delta y}{\Delta t} \approx 2 - \frac{y}{10 + t}.
\]
Of course,
\[
\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt},
\]
and it appears that the error in our approximation (recall that we assumed that the concentration of the water in the tank stayed constant during the time interval \(\Delta t\)) goes to zero as \(\Delta y \to 0\), so we say
\[
\frac{dy}{dt} = 2 - \frac{y}{10 + t} \quad \text{with} \quad y(0) = 1.
\]

This is a first order linear differential equation. Following procedure, we rewrite it as \(\frac{dy}{dt} + y\frac{1}{10 + t} = 2\) and let
\[
\begin{array}{|c|c|}
\hline
P & 1 \\
\hline
Q & 10 + t \\
\hline
\end{array}
\]
So according to our method,
\[
y = e^{-P} \int e^P Q \, dt = \frac{1}{10 + t} \int 2(10 + t) \, dt = \frac{1}{10 + t} (20t + t^2 + C).
\]
Since $y(0) = 1$, we have $1 = \frac{1}{10 + 0}(20(0) + (0)^2 + C)$, so $C = 10$, and the equation solving this initial value problem (and thus giving the amount of salt in the tank at time $t$) is

$$y = \frac{1}{10 + t}(20t + t^2 + 10).$$

□

Problem 5. (10pts) Solve the system of equations:

$$x_1 + 2x_2 + 2x_3 = 2$$
$$3x_1 + 7x_2 + 8x_3 = 8$$
$$x_1 + 3x_2 + 4x_3 = 4$$

Solution. We begin by writing this system in terms of an augmented matrix:

$$
\begin{bmatrix}
1 & 2 & 2 & | & 2 \\
3 & 7 & 8 & | & 8 \\
1 & 3 & 4 & | & 4
\end{bmatrix}
$$

which we then row reduce using the indicated elementary row operations:

$$
\begin{bmatrix}
1 & 2 & 2 & | & 2 \\
3 & 7 & 8 & | & 8 \\
1 & 3 & 4 & | & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 2 & | & 2 \\
0 & 1 & 2 & | & 2 \\
0 & 1 & 2 & | & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 2 & | & 2 \\
0 & 1 & 2 & | & 2 \\
0 & 0 & 0 & | & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -2 & | & -2 \\
0 & 1 & 2 & | & 2 \\
0 & 0 & 0 & | & 0
\end{bmatrix}
$$

from which we easily read the solutions: $x_3 = t$, $x_2 = 2 - 2t$, $x_1 = -2 + 2t$.

□

Problem 6. (10pts) Let $A$ be the $5 \times 5$ matrix

$$A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 & 15 \\
16 & 17 & 18 & 19 & 20 \\
21 & 22 & 23 & 24 & 25
\end{bmatrix}.$$

It is a fact that

$$A \begin{bmatrix}
0 \\
8 \\
2 \\
3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.$$
(a–5pts) How many solutions does the homogeneous system of equations

\[
\begin{align*}
7x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 0 \\
6x_1 + 7x_2 + 8x_3 + 9x_4 + 10x_5 &= 0 \\
11x_1 + 12x_2 + 13x_3 + 14x_4 + 15x_5 &= 0 \\
16x_1 + 17x_2 + 18x_3 + 19x_4 + 20x_5 &= 0 \\
21x_1 + 22x_2 + 23x_3 + 24x_4 + 25x_5 &= 0
\end{align*}
\]

have (and how can you tell, of course)?

**Solution.** We know that a system of equations has no solutions, exactly one solution, or infinitely many solutions. This system has at least 2 solutions: the trivial solution which every homogeneous system has, and the solution \( x_1 = 0, x_2 = 8, x_3 = -13, x_4 = 2, x_5 = 3 \) which we can read off of the matrix equation representing this system (as given above). Thus we conclude that this system has infinitely many solutions. \( \square \)

(b–5pts) Is \( A \) invertible or not (and how can you tell, of course)?

**Solution.** One of our theorems says that an \( n \times n \) matrix \( B \) is invertible if and only if \( B \vec{x} = \vec{0} \) has only the trivial solution. Since \( A \vec{x} = \vec{0} \) has more than just the trivial solution (in particular, \( \vec{x} = \begin{bmatrix} 0 \\ 8 \\ -13 \\ 2 \\ 3 \end{bmatrix} \) is a solution), we conclude that \( A \) is not invertible.

**Extra Credit (3pts).** Your favorite invertible matrix is

\[
A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \\ 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 \\ 31 & 32 & 33 & 34 & 35 & 36 \end{bmatrix}
\]

One night, you forget to lock your bedroom door, and while you are sleeping, your little brother comes and does a bunch of elementary row operations on \( A \). The next morning, you find the following two matrices on your desk:

\[
B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

One comes from your beloved matrix \( A \), while the other is just some matrix which your brother left lying around. Explain how you can tell which is which.

**Solution.** We know from the text that elementary row operations are reversible: that is, if I can transform a matrix \( D \) into \( E \) using elementary row operations, then I can transform \( E \) back to \( D \) using elementary row operations. So, if \( B \) was obtained by doing row operations on \( A \), then one can obtain \( A \) and hence \( I_6 \) by doing row operations on \( B \) (here we used one of our theorems which says that any invertible matrix is row equivalent to the identity). We conclude that if \( B \) was obtained from \( A \), then \( B \) is row equivalent to the identity matrix. Even a cursory glance at \( B \), however, shows that this can’t be true. No elementary row operations are able to obtain leading entries in the last two diagonal spots while keeping those higher up the diagonal. To be completely precise, observe that

\[
B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 + 0 + 0 + 0 + 0 + 0 - 1 \\ 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]
so that the system $B \vec{x} = \vec{0}$ has more than just the trivial solution, and hence (by one of our theorems) cannot be row equivalent the the identity matrix. We conclude that $C$ was obtained from your favorite matrix, and not $B$. □