Problem 1. (20pts) Consider the sequence \( \left\{ \frac{2 + \sqrt{n}}{n} \right\}_{n=1}^{\infty} \).

(a–10pts) Does this sequence converge or diverge?

Solution. Note that the function \( f(x) = \frac{2 + \sqrt{x}}{x} \) is continuous on \([1, \infty)\) and of course \( f(n) = \frac{2 + \sqrt{n}}{n} \), so to show that the sequence converges, we only need to show that \( \lim_{x \to \infty} f(x) \) exists. Of course, the form of \( \lim_{x \to \infty} f(x) \) is \( \frac{\infty}{\infty} \), and the derivative of the top and bottom exist and are nonzero, so by L'Hospital's Rule,

\[
\lim_{x \to \infty} \frac{2 + \sqrt{x}}{x} = \lim_{x \to \infty} \frac{\frac{1}{2\sqrt{x}}}{1} = \lim_{x \to \infty} \frac{1}{2\sqrt{x}} = 0.
\]

We conclude that the sequence converges. □

(b–10pts) Does the series \( \sum_{n=1}^{\infty} \frac{2 + \sqrt{n}}{n} \) converge or diverge?

Solution. Note that \( 0 \leq \frac{1}{n} \leq \frac{2}{n} \leq \frac{2 + \sqrt{n}}{n} \) for all \( n \geq 1 \) since \( \sqrt{n} \geq 0 \). Thus by the comparison test, it is enough to show that \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges. But this follows immediately from the P test, and we are finished. □

Problem 2. (10pts) Do one of the following two problems.

(a–10pts) Decide if \( \sum_{n=1}^{\infty} \frac{(-1)^n}{e^n} \) converges or diverges.

Solution. Since \( \frac{n}{e^n} \) is always positive, the series clearly alternates. By the alternating series test, we will know the series converges if \( \{\frac{n}{e^n}\} \to 0 \) and \( \{\frac{n}{e^n}\} \) is decreasing. We demonstrate both these conditions by considering the function \( f(x) = \frac{x}{e^x} \). Of course, \( f(n) = \frac{n}{e^n} \), so it is enough to show that \( f(x) \) is decreasing on \([1, \infty)\) and \( \lim_{x \to \infty} f(x) = 0 \). The former follows since \( f'(x) = \frac{e^x - xe^x}{(e^x)^2} = 1 - \frac{e^x}{e^x} < 0 \) for \( x \geq 1 \), and the latter is true because \( \lim_{x \to \infty} = \frac{x}{e^x} \lim_{x \to \infty} \frac{1}{e^x} = 0 \) (using L'Hospital's rule and the fact that \( \lim_{x \to \infty} \frac{x}{e^x} \) has form \( \frac{\infty}{\infty} \)). □

(b–10pts) Decide if \( \sum_{n=1}^{\infty} \frac{n}{e^n} \) converges or diverges.

Solution. Note that the function \( f(x) = \frac{x}{e^x} \) is positive and continuous on the positive real numbers. Furthermore, since \( f'(x) = \frac{e^x - xe^x + 2x}{(e^x)^2} = \frac{1 - 2x^2}{e^x} < 0 \) for \( x \geq 1 \), we see that \( f(x) \) is decreasing. Thus, by the integral test, it
is enough to decide whether \( \int_{1}^{\infty} f(x) \, dx \) converges or diverges. Here

\[
\int_{1}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{1}^{t} xe^{-x^{2}} \, dx = \lim_{t \to \infty} \int_{x=1}^{x=t} e^{u} \, \frac{du}{2}
\]

\[
= \lim_{t \to \infty} e^{u} \bigg|_{x=1}^{x=t} = \lim_{t \to \infty} \left( \frac{1}{2} - \frac{t^{2}}{2e^{t}} + \frac{1}{2e} \right) = \frac{1}{2e}.
\]

Note the \( u \)-substitution \( u = -x^{2} \) with \( du = \frac{du}{2} = x \, dx \). We conclude that the integral, and hence the series converges.

\[\square\]

**Problem 3. (10pts) Do one of the following two problems.**

(a–10pts) Decide if \( \sum_{n=1}^{\infty} \frac{n}{e^{2n}} \) converges or diverges.

**Solution.** Note that \( \lim_{n \to \infty} \sqrt[n]{\frac{n}{e^{2n}}} = \lim_{n \to \infty} \left( \frac{n}{e^{2}} \right)^{1/n} = \lim_{n \to \infty} \frac{n}{e^{2}} = \infty \). Thus by the root test, the series diverges.

\[\square\]

(b–10pts) Decide if \( \sum_{n=2}^{\infty} \left( 1 + \frac{1}{\ln n} \right) \frac{1}{n^{2}} \) converges or diverges.

**Solution.** Note that \( \lim_{n \to \infty} \left( 1 + \frac{1}{\ln n} \right) \frac{1}{n^{2}} = \lim_{n \to \infty} \left( 1 + \frac{1}{\ln n} \right) = 1 \). The limit comparison test says that if \( \lim_{n \to \infty} \frac{a_{n}}{b_{n}} > 0 \), then either both \( \sum_{n=1}^{\infty} a_{n} \) and \( \sum_{n=1}^{\infty} b_{n} \) converge or both diverge. So either both \( \sum_{n=2}^{\infty} \left( 1 + \frac{1}{\ln n} \right) \frac{1}{n^{2}} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^{2}} \) converge or both diverge. Of course, we know that the latter converges by the P-test, so we conclude that \( \sum_{n=2}^{\infty} \left( 1 + \frac{1}{\ln n} \right) \frac{1}{n^{2}} \) converges as well.

\[\square\]

**Problem 4. (10pts) Do one of the following two problems:**

(a–10pts) Find a power series and radius of convergence for the function \( f(x) = \frac{x}{8 - x^{3}} \).

**Solution.** Note that

\[
\frac{x}{8 - x^{3}} = \frac{x}{8} \left( 1 + \frac{1}{x^{3}/8} \right) = \frac{x}{2} \left( 1 - \left( \frac{x}{2} \right) \right) ^{3}.
\]

Of course,

\[
\frac{1}{1 - \left( \frac{x}{2} \right)^{3}} = \sum_{n=0}^{\infty} \left( \frac{x}{2} \right)^{3n} = \sum_{n=0}^{\infty} \frac{x^{3n}}{2^{3n}}
\]

whenever \( |\left( x/2 \right)^{3}| < 1 \), that is, whenever \( |x| < 2 \) since it is geometric. Then

\[
\frac{x}{2} \left( 1 - \frac{1}{1 - \left( \frac{x}{2} \right)^{3}} \right) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{x^{3n}}{2^{3n}} = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{2^{3n+3}}
\]

whenever \( |x| < 2 \) since we may move the constant \( x/8 \) past the summation. The radius of convergence here is thus \( R = 2 \).

\[\square\]

(b–10pts) Write the Maclaurin Series for \( e^{3x} \) (you may not use your card, i.e., you may not use the fact that you know a power series for \( e^{x} \)).

**Solution.** We know that the Maclaurin series for \( f(x) \) is \( \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^{n} \), so we are finished if we can find \( f^{(n)}(0) \). Here we see that:
Thus the Maclaurin series for \( e^{3x} \) is

\[
\sum_{n=0}^{\infty} \frac{3^n}{n!} x^n.
\]

**Problem 5.** (15pts) Let \( f(x) = \sin x \).

(a−5pts) Use the Taylor Polynomial of degree 5 to estimate \( \sin(1/2) \) (you may use your card and you need not simplify).

**Solution.** We know that the Taylor series for \( \sin x \) is

\[
\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},
\]

so the degree 5 Taylor polynomial is

\[
T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.
\]

Thus we have

\[
\sin(1/2) \approx T_5(1/2) = (1/2) - \frac{(1/2)^3}{3!} + \frac{(1/2)^5}{5!}.
\]

(b−10pts) Is your estimate within 1/100 of \( \sin(1/2) \)?

**Solution.** We know that error is equal to \( |\sin(x) - T_5(x)| \) and that for \( |x| < R \),

\[
|\sin(x) - T_5(x)| = |R_5| \leq \frac{M|x|^6}{6!}
\]

where \( M \) is such that \( |f^{(6)}(x)| \leq M \) on \( |x| < R \). Of course, \( f^{(6)}(x) = -\sin x \), and \( |\sin x| \leq 1 \) for all \( x \), so \( M = 1 \) works. Taking \( R = 1 \) (note that 1/2 < 1, so that for \( R = 1 \), the bound \( |\sin(x) - T_5(x)| = |R_5| \leq \frac{M|x|^6}{6!} \) for \( |x| < R \) holds when \( x = 1/2 \), we have

\[
|\sin(1/2) - T_5(1/2)| = |R_5| \leq \frac{1 \cdot 1^6}{6!} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \leq \frac{1}{100}.
\]

We conclude that our estimate is within 1/100 of \( \sin(1/2) \).