Problem 1.  (10pts) For your senior project you decide to design a crock-pot for cooking up your homemade energy drink (which you call Calculus Monster). The temperature in the crock-pot $t$ minutes after turning it on is $T(t) = \frac{6t^2 + e^t}{2t^3 + e^t}$ degrees Fahrenheit. What is the average temperature in the crock-pot from time $t = 2$ to time $t = 8$?

Solution. The formula for the average value of $f(x)$ from $x = a$ to $x = b$ is \[
\frac{1}{b-a} \int_a^b f(x) \, dx.
\] In this instance, we have

\[
\int_0^6 \frac{6t^2 + e^t}{2t^3 + e^t} \, dt = \int_0^6 \frac{1}{u} \, du = \ln |u| \bigg|_0^6 = \ln(2(6)^3 + e^6) - \ln(2(0) + e^0) = \ln(2(6)^3 + e^6) \text{ degrees}.
\]

We have used the $u$-substitution

Let $u = 2t^3 + e^t$  
so $\frac{du}{dt} = 6t^2 + e^t$  
and $du = (6t^2 + e^t) \, dt$.

\[
\blacksquare
\]

Problem 2.  (10pts) You decide to make your mother a sculpture for mother’s day (out of titanium, of course). Your design calls for the object obtained by rotating the area bounded by $f(x) = x^2 + 5$, $g(x) = \frac{e^x}{x}$, $x = 1$, and $x = 4$ about the $y$-axis (where $x$ and $y$ are in meters). What is the volume of titanium you will require?

Solution. We use the method of shells since we want our boxes to be vertical, which is parallel to the axis of rotation, the $y$-axis. In general, the formula for computing volume by shells is

\[
\int_a^b 2\pi \text{ (radius)} \times \text{ (height)} \, dx.
\]

Here the radius is simply $x$, the top function is $f(x)$ and the bottom function is $g(x)$, so the height is $f(x) - g(x) = x^2 + 5 - \frac{e^x}{x}$ and we have

\[
\int_1^4 2\pi x \left( x^2 + 5 - \frac{e^x}{x} \right) \, dx = 2\pi \int_1^4 (x^3 + 5x - e^x) \, dx
\]
\[
= 2\pi \left( \frac{x^4}{4} + \frac{5}{2} x^2 - e^x \right) \int_1^4 = 2\pi \left( \frac{4^4}{4} + \frac{5}{2} 4^2 - e^4 \right) = 2\pi \left( \frac{1}{4} + \frac{5}{2} - e^1 \right)
\]
cubic feet of titanium.

**Problem 3.** (10pts) A 5 foot rope which weighs 10 lbs is laying on the floor. Compute the work required to lift one end of the rope 5 feet into the air.

**Solution.** We proceed with the usual method.

- Cut the rope into \( n \) pieces of width \( \Delta x = \frac{5}{n} \).
- Label the endpoints \( x_i \) from the end to be lifted.
- Note that the rope weighs 2 pounds per foot, and thus the weight of \([x_{i-1}, x_i]\) is \(2\Delta x\) (the linear density of the rope times length). Note also that the point \( x_i \) ends up \(5 - x_i\) feet off the floor. Thus, assuming that the mass of the \( i \)th piece is clumped at \( x_i \), the work to lift the \( i \)th piece to its end position is approximately force times distance which equals weight times distance, or \(2\Delta x(5 - x_i)\).
- The work to lift the whole rope is thus approximately \( \sum_{i=1}^{n} 2(5 - x_i)\Delta x \).
- Since our approximating seems to improve as \( n \) gets larger (because the distance from \( x_{i-1} \) to \( x_i \) gets shorter), we believe that work equals \( \lim_{n \to \infty} \sum_{i=1}^{n} 2(5 - x_i)\Delta x \). Note that this limits to \( \sum_{i=1}^{n} (5 - x_i) \int_0^5 (5 - x) \, dx = 2 \left( \frac{5x - x^2}{2} \right) \bigg|_0^5 = 2(25 - 25/2) = 25 \) slugs.

**Problem 4.** (10pts) You are an avid seashell hunter (it all began with your mom—she sells sea shells down by the seashore). Suppose that \( g(x) \) computes the number of hours it takes you to find \( x \) sea shells.

(a) In a sentence or two, explain what \((g^{-1})'(x)\) computes.

**Solution.** Since \( g(x) \) computes the number of hours it takes to find \( x \) seashells, \( g^{-1}(x) \) computes the number of seashells you have found after \( x \) hours. Thus \((g^{-1})'(x)\) computes the rate of change of the number of seashells with respect to hours at hour \( x \), that is, the instantaneous speed at which you are finding seashells (in seashells per hour) after \( x \) hours.

(b) Compute \((g^{-1})'(3)\) if \( g(4) = 3, g(3) = 7, g'(3) = 10, g'(4) = 12 \) and \( g'(7) = 4 \).

**Solution.** We use the formula \((g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))}\). Note that \( g(4) = 3 \) so that \( g^{-1}(3) = 4 \). Then

\[
(g^{-1})'(3) = \frac{1}{g'(g^{-1}(3))} = \frac{1}{g'(4)} = \frac{1}{12}
\]
seashells per hour.

**Problem 5.** (10pts) Compute \( \frac{d}{dx} \left( \frac{(\arcsin x)^2(x+1)^2}{(x+2)^3} \right) \).

**Solution.** We use logarithmic differentiation. Since

\[
y = \frac{(\arcsin x)^2(x+1)^2}{(x+2)^3},
\]
we have that

\[
\ln y = \ln \left( \frac{(\arcsin x)^2(x+1)^2}{(x+2)^3} \right) = \ln(\arcsin x)^2 + \ln(x+1)^2 - \ln(x+2)^3 = 2\ln(\arcsin x) + x \ln(x+1) - 3 \ln(x+2),
\]
where we have used the usual rules for simplifying logarithmic functions.

Now differentiating both sides yields:

\[
y' = 2 \cdot \frac{1}{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}} + \left( \ln(x+1) + x \frac{1}{x+1} \right) - 3 \frac{1}{x+2}
\]
where we used that \( \frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)} \) as well as the product rule. Solving for \( y' \) gives:

\[
y' = y \left( \frac{2}{\sqrt{1-x^2} \arcsin x} + \ln(x+1) + \frac{x}{x+1} - \frac{3}{x+2} \right)
\]

\[
= \left( \frac{(\arcsin x)^2 (x+1)^3}{(x+2)^3} \right) \left( \frac{2}{\sqrt{1-x^2} \arcsin x} + \ln(x+1) + \frac{x}{x+1} - \frac{3}{x+2} \right)
\]

\[\Box\]

**Problem 6.** (10 pts) Each of the entries on the right corresponds with exactly one entry on the left (after a \( u \)-substitution, evaluating an integral, taking a derivative, or some other manipulation). Match the entries. This is the only problem on the test for which you need show no work.

<table>
<thead>
<tr>
<th>( E )</th>
<th>( \int \frac{\sec^2 x}{\sqrt{\tan^2 x - 1}} , dx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )</td>
<td>( \int \frac{-2x}{\sqrt{1-x^2}} , dx )</td>
</tr>
<tr>
<td>( A )</td>
<td>( \int \frac{1}{\sqrt{4-x^2}} , dx )</td>
</tr>
<tr>
<td>( C )</td>
<td>( \int \frac{x}{\sqrt{1-x^4}} , dx )</td>
</tr>
<tr>
<td>( H )</td>
<td>( \int \frac{1}{x\sqrt{4x^2 - 16}} , dx )</td>
</tr>
<tr>
<td>( J )</td>
<td>( f^{-1}(x) ) if ( f(x) = \tan \frac{x}{3} )</td>
</tr>
<tr>
<td>( G )</td>
<td>( \int \frac{3}{1+9x^2} , dx )</td>
</tr>
<tr>
<td>( B )</td>
<td>( (f')^{-1}(x) ) if ( f(x) = 3^x )</td>
</tr>
<tr>
<td>( D )</td>
<td>( (f^{-1})'(x) ) if ( f(x) = 3^x )</td>
</tr>
<tr>
<td>( I )</td>
<td>( f^{-1}(x) ) if ( f(x) = 3^x )</td>
</tr>
</tbody>
</table>

| \( A \) | \( \int \frac{1}{\sqrt{1-u^2}} \, du \) |
| \( B \) | \( \log_3(x) - \frac{\ln \ln 3}{\ln 3} \) |
| \( C \) | \( \frac{1}{2} \sin^{-1}(x^2) + C \) |
| \( D \) | \( \frac{1}{x \ln 3} \) |
| \( E \) | \( \cosh^{-1}(\tan x) + C \) |
| \( F \) | \( \int \frac{1}{\sqrt{u}} \, du \) |
| \( G \) | \( \tan^{-1}(3x) + C \) |
| \( H \) | \( \frac{1}{4} \sec^{-1} \left( \frac{x}{2} \right) + C \) |
| \( I \) | \( \log_3(x) \) |
| \( J \) | \( \tan^{-1}(3x) \) |