A few integrals:

1. \[ \int \sec^3(u) \, du = \frac{1}{2} \tan(u) \sec(u) + \frac{1}{2} \ln |\sec(u) + \tan(u)| + C \]
2. \[ \int \frac{\sqrt{2au - u^2}}{u} \, du = \sqrt{2au - u^2} + a \cos^{-1} \left( \frac{a - u}{a} \right) + C \]
3. \[ \int \frac{\sqrt{a + bu}}{u} \, du = 2\sqrt{a + bu} + \sqrt{a} \ln \left| \frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}} \right| + C \text{ if } a > 0 \]
4. \[ \int u \cos^{-1} u \, du = \frac{2u^2 - 1}{4} \cos^{-1} u - \frac{u \sqrt{1 - u^2}}{4} + C \]

Problem 1. (10pts) Compute the following limit: \( \lim_{x \to \infty} \left( 1 + \frac{2}{x} \right)^x \).

Solution. Note that

\[ \lim_{x \to \infty} \ln \left( 1 + \frac{2}{x} \right)^x = \lim_{x \to \infty} x \ln \left( 1 + \frac{2}{x} \right) = \lim_{x \to \infty} \frac{\ln(1 + \frac{2}{x})}{\frac{1}{x}} \]

is of the form \( \frac{0}{0} \) since

\[ \ln \left( 1 + \frac{2}{x} \right) \to \ln 1 = 0 \]
as \( x \to \infty \) and that all the derivatives are well behaved, so we may use L'Hospital's rule. Thus

\[ \lim_{x \to \infty} \frac{\ln(1 + \frac{2}{x})}{\frac{1}{x}} = \lim_{x \to \infty} \frac{-\frac{2}{x^2}}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{2}{2} = 2, \]

where we have used that \( \frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)} \). We conclude that

\[ \lim_{x \to \infty} \left( 1 + \frac{2}{x} \right)^x = e^{\lim_{x \to \infty} \ln(1 + \frac{2}{x})} = e^{\frac{2}{1}} = e^2. \]

Problem 2. (10pts) Compute \( \int_0^1 \frac{x^3}{\sqrt{1 + x^2}} \, dx \).

Solution. Let \( x = \tan \theta \) with the lie \( dx = \sec^2 \theta \, d\theta \). Then we have

\[ \int_0^1 \frac{x^3}{\sqrt{1 + x^2}} \, dx = \int_{x=0}^{\theta=1} \frac{\tan^3 \theta}{\sqrt{1 + \tan^2 \theta}} \sec^2 \theta \, d\theta \]

\[ = \int_{x=0}^{\theta=1} \tan^3 \theta \sec^2 \theta \, d\theta = \int_{x=0}^{\theta=1} \tan^3 \theta \sec \theta \frac{\sec \theta}{\sec \theta} \, d\theta = \int_{x=0}^{\theta=1} \tan^3 \theta \sec \theta \, d\theta \]

\[ = \int_{x=0}^{\theta=1} (\tan^2 \theta) \tan \theta \sec \theta \, d\theta = \int_{x=0}^{\theta=1} (\sec^2 \theta - 1) \tan \theta \sec \theta \, d\theta. \]
Now do the $u$-substitution $u = \sec \theta$ with the fib that $du = \sec \theta \tan \theta \, d\theta$. We get

$$
\int_{x=0}^{x=1} (\sec^2 \theta - 1) \tan \theta \sec \theta \, d\theta = \int_{x=0}^{x=1} (u^2 - 1) \, du = \left( \frac{u^3}{3} - u \right) \bigg|_{x=0}^{x=1} = \left( \frac{\sec^3 \theta}{3} - \sec \theta \right) \bigg|_{x=0}^{x=1}.
$$

Using the usual picture,

we see that $\sec \theta = \sqrt{1 + x^2}$ yielding

$$
\left( \frac{\sec^3 \theta}{3} - \sec \theta \right) \bigg|_{x=0}^{x=1} = \left( \frac{(1 + x^2)^{3/2}}{3} - \sqrt{1 + x^2} \right) \bigg|_{0}^{1} = \left( \frac{(2)^{3/2}}{3} - \sqrt{2} \right) - \left( \frac{1}{3} - 1 \right) = \frac{2^{3/2}}{3} - \sqrt{2} + \frac{2}{3}.
$$

\[
\square
\]

**Problem 3.** (10pts) Compute $\int \frac{x}{(x+1)(x+2)} \, dx$.

**Solution.** We do expansion by partial fractions. Write

$$
x = \frac{A}{x+1} + \frac{B}{x+2},
$$

then clear denominators to obtain the equation

$$
x = A(x+2) + B(x+1) = (A+B)x + (2A + B)
$$

which in turn yields the system of equations:

$$
A + B = 1 \\
2A + B = 0.
$$

Subtracting the first equation from the second gives $A = -1$, whence we conclude by the first equation that $B = 2$. Thus

$$
\int \frac{x}{(x+1)(x+2)} \, dx = \int \left( \frac{-1}{x+1} + \frac{2}{x+2} \right) \, dx = -\ln |x+1| + 2 \ln |x+2| + C
$$

where we used the fact that $\int \frac{1}{x+a} \, dx = \ln |x+a| + C$ for any constant $a$.  

\[
\square
\]

**Problem 4.** (10pts) Compute $\int \limits_{0}^{1} e^{2\sqrt{x}} \, dx$.

**Solution.** Let $u = 2\sqrt{x}$, so that $\frac{du}{dx} = \frac{1}{\sqrt{x}}$ and thus the lie is that $\frac{u}{2} \, du = \sqrt{x} \, dx = dx$. Also, $u = 0$ when $x = 0$ while $u = 2$ when $x = 1$, so we have

$$
\int_{x=0}^{x=1} e^{2\sqrt{x}} \, dx = \int_{0}^{2} \frac{u}{2} e^{u} \, du = \frac{1}{2} \int_{0}^{2} u e^{u} \, du.
$$

Now proceed with integration by parts. Let $f(u) = u$ and $g'(u) = e^{u}$ giving the table

\[
\begin{array}{c|c|c}
 f(u) & u & e^{u} \\
 g(u) & 1 & e^{u} \\
 f'(u) & 1 & e^{u}
\end{array}
\]

and

$$
\frac{1}{2} \int_{0}^{2} u e^{u} \, du = \frac{1}{2} \int_{0}^{2} f(u)g'(u) \, du = \frac{1}{2} \left[ f(u)g(u) \right]_{0}^{2} - \frac{1}{2} \int_{0}^{2} f'(u)g(u) \, du
$$

\[
\square
\]
Problem 5. (15pts) Estimate \( \int_0^1 \cos(x^2) \, dx \) using the Midpoint rule with error at most \( 1/100 \). (You need not simplify).

Solution. The Midpoint rule says that \( \int_a^b f(x) \, dx \approx \sum_{i=1}^n f(x_i) \Delta x \) where

\[
\Delta x = \frac{b-a}{n}, \quad x_i = a + i \Delta x, \quad x_i = \frac{x_{i-1} + x_i}{2}
\]

with

\[
E_{M_n} = \left| \int_a^b f(x) \, dx - M_n \right| \leq \frac{K(b-a)^3}{24n^2}
\]

where \( K \) is such that \( K \geq |f''(x)| \) on \([a, b]\).

So we need to find an \( n \) such that

\[
E_{M_n} \leq \frac{K(b-a)^3}{24n^2} \leq 1/100.
\]

Observe that if \( f(x) = \cos(x^2) \), then

\[
\begin{align*}
f'(x) &= -2x \sin(x^2) \\
f''(x) &= -2 \sin(x^2) - 4x^2 \cos(x^2).
\end{align*}
\]

Since \( x^2, \sin(x^2), \) and \( \cos(x^2) \) are all positive on \([0,1]\), we have

\[
-2 \sin(x^2) - 4x^2 \cos(x^2) \leq 0
\]

for all \( x \in [0,1] \). Furthermore, \( \sin(x^2), x^2, \) and \( \cos(x^2) \) are all bounded above by 1 on \([0,1]\), so

\[
-6 \leq -2 \sin(x^2) - 4x^2 \cos(x^2) \leq 0
\]

for all \( x \in [0,1] \), and taking \( K = 6 \), we have \( K \geq |f''(x)| \) on \([0,1]\).

Thus we need \( n \) such that

\[
\frac{6(1-0)^3}{24n^2} \leq \frac{1}{100},
\]

that is, such that \( \frac{100}{4} \leq n^2 \), so \( 25 \leq n^2 \), or \( 5 \leq n \). We take \( n = 5 \).

Now fall off the horse. We have

\[
\begin{align*}
\Delta x &= \frac{1}{5} \\
x_i &= i \frac{1}{5} \\
x_i &= \frac{2i - 1}{10},
\end{align*}
\]

so

\[
\int_0^1 \cos(x^2) \, dx \approx \sum_{i=1}^5 \cos(x_i^2) \Delta x
\]

\[
= \cos\left( \left( \frac{1}{10} \right)^2 \right) \frac{1}{5} + \cos\left( \left( \frac{3}{10} \right)^2 \right) \frac{1}{5} + \cos\left( \left( \frac{5}{10} \right)^2 \right) \frac{1}{5} + \cos\left( \left( \frac{7}{10} \right)^2 \right) \frac{1}{5} + \cos\left( \left( \frac{9}{10} \right)^2 \right) \frac{1}{5}
\]

whatever that turns out to be, and the error is \( \leq 1/100 \).

Just because it is interesting, note that Mathematica says that
\[
\cos \left( \left( \frac{1}{10} \right)^2 \right) \frac{1}{5} + \cos \left( \left( \frac{3}{10} \right)^2 \right) \frac{1}{5} + \cos \left( \left( \frac{5}{10} \right)^2 \right) \frac{1}{5} + \cos \left( \left( \frac{7}{10} \right)^2 \right) \frac{1}{5} + \cos \left( \left( \frac{9}{10} \right)^2 \right) \frac{1}{5} = 0.90732\ldots
\]

while
\[
\int_0^1 \cos(x^2) \, dx = 0.904524\ldots,
\]
so the error is actually 0.002805\ldots which is \( \leq 1/100 \).

**Problem 6.** (10pts) Does \( \int_1^\infty \frac{(\cos(\sin(\cos(\sin(x)))))^2}{x^2 + x} \, dx \) converge or diverge?

**Solution.** Observe that
\[
0 \leq (\cos(\sin(\cos(\sin(x)))))^2
\]
for all \( x \geq 1 \) because of the square. Furthermore, since \( \cos(x) \leq 1 \) for all \( x \geq 1 \) we conclude that
\[
\cos(\sin(\cos(\sin(x)))) \leq 1
\]
and hence that
\[
(\cos(\sin(\cos(\sin(x)))))^2 \leq 1
\]
for all \( x \geq 1 \). Thus
\[
0 \leq \frac{(\cos(\sin(\cos(\sin(x)))))^2}{x^2 + x} \leq \frac{1}{x^2}
\]
for all \( x \geq 1 \) (because the denominator got smaller while the numerator got larger). Therefore, by the comparison test,
\[
\int_1^\infty \frac{(\cos(\sin(\cos(\sin(x)))))^2}{x^2 + x} \, dx
\]
converges if \( \int_1^\infty \frac{1}{x^2} \, dx \) converges. But we know that \( \int_1^\infty \frac{1}{x^2} \, dx \) converges by the P-test. We conclude that the integral in question converges. \( \square \)