Problem 1. (10pts) Suppose that the function \( f(t) = \frac{100}{1 + t^2} \) gives the temperature of your car’s engine in degrees Fahrenheit \( t \) minutes after you shut it off (at the end of a long drive; by the way, you live in Antarctica). What is the average temperature of your car’s engine during the first 10 minutes after shutting it off?

Solution. The formula for the average value of \( f(x) \) from \( x = a \) to \( x = b \) is \( \frac{1}{b-a} \int_a^b f(x) \, dx \). In this instance, we have

\[
\frac{1}{10-0} \int_0^{10} \frac{100}{1 + t^2} \, dt = \frac{100}{10} \int_0^{10} \frac{1}{1 + t^2} \, dt = 10 \tan^{-1}(t) \bigg|_0^{10} = 10(\tan^{-1}(10) - \tan^{-1}(0)) = 10 \tan^{-1}(10)
\]

degrees Fahrenheit.

Problem 2. (10pts) Find the slope of the tangent line to the function \( y = (\sin x)^{\cos x} \) at the point \( x = \pi/6 \). (Hint: use logarithmic differentiation; oh, and don’t simplify).

Solution. The slope of the tangent is given by the derivative of \( y = (\sin x)^{\cos x} \) at \( x = \pi/6 \). Note that

\[
\ln y = \ln(\sin x)^{\cos x} = \cos x \ln \sin x,
\]

and differentiating both sides yields

\[
\frac{y'}{y} = -\sin x \ln \sin x + \cos x \frac{\cos x}{\sin x},
\]

where we used the product rule, and that fact that \( \frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)} \). Thus

\[
y' = y \left( -\sin x \ln \sin x + \frac{\cos^2 x}{\sin x} \right) = (\sin x)^{\cos x} \left( -\sin x \ln \sin x + \frac{\cos^2 x}{\sin x} \right),
\]

and the slope of \( y \) at \( x = \pi/6 \) is

\[
(\sin(\pi/6))^{\cos(\pi/6)} \left( -\sin(\pi/6) \ln(\pi/6) + \frac{\cos^2(\pi/6)}{\sin(\pi/6)} \right).
\]

Problem 3. (10pts) A certain 10 foot, weightless chain dangles off a cliff (above swirling waters and treacherous rocks at the cliff’s foot). A weightless bucket hangs at the end of the chain. The bucket initially contains 2 gallons of coffee weighing 2 pounds per gallon. With fervent anticipation, you pull the bucket up at a constant speed, but unfortunately, the bucket leaks at a constant rate, and just finishes draining as it reaches the top of the cliff. Give a Riemann sum which computes the work done lifting the coffee.

Solution. We follow the usual format.

1. Cut the chain (or rather, the path of the bucket) into \( n \) pieces of length \( \Delta x = \frac{10}{n} \).
2. Label the endpoints of the pieces \( x_i = i\Delta x \) from the bottom.
Problem 4. (10pts) Compute the volume of the solid obtained by rotating the region bounded by $y = 0$, $y = e^x$, $x = -1$ and $x = 1$ about the line $y = -1$.

Solution. We use the method of washers since we want our boxes to be vertical, which is perpendicular to the axis of rotation, the line $y = -1$.

In general, the formula for computing volume by washers is

$$\int_a^b \pi((\text{outer radius})^2 - (\text{inner radius})^2) \, dx.$$ 

Here the outer radius is $e^x + 1$, the distance from the top of the graph to the axis of rotation, and the inner radius is 1, the distance from the bottom of the region (i.e., the line $y = 0$) to the axis of rotation. We have

$$\int_{-1}^{1} \pi((e^x + 1)^2 - 1^2) \, dx = \pi \int_{-1}^{1} (e^{2x} + 2e^x + 1 - 1) \, dx = \pi \int_{-1}^{1} (e^{2x} + 2e^x) \, dx$$

$$= \pi \left( \frac{e^{2x}}{2} + 2e^x \right) \bigg|_{-1}^{1} = \pi \left( \left( \frac{e^2}{2} + 2e \right) - \left( \frac{e^{-2}}{2} + 2e^{-1} \right) \right).$$

Note that we used a $u$-substitution to integrate $e^{2x}$. In particular, taking $u = 2x$ so that the lie is $\frac{du}{2} = dx$, we get

$$\int_{-1}^{1} e^{2x} \, dx = \int_{x=-1}^{x=1} \frac{e^{2u}}{2} \, du = \frac{e^u}{2} \bigg|_{x=-1}^{x=1} = \frac{e^2}{2} - \frac{1}{2} = \frac{e^2 - 1}{2}.$$ 

Problem 5. (10pts) Suppose that $g(x) = f^{-1}(x)$, $G(x) = 2^{g(x)}$, $f(3) = 4$, $f'(3) = 8$, $f(4) = 10$, and $f'(10) = 12$. What is $G'(4)$?

Solution. Recall that $\frac{d}{dx}(a^x) = a^x \ln a$, and by the chain rule that $\frac{d}{dx}(af(x)) = a f'(x)(\ln a) f'(x)$. Hence

$$G'(x) = 2^{g(x)}(\ln 2)g'(x),$$ 

and we want to compute

$$G'(4) = 2^{g(4)}(\ln 2)g'(4).$$

Now $f(3) = 4$, so $g(4) = 3$, and by our formula $g'(x) = \frac{1}{f'(g(x))}$, so that $g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(3)} = \frac{1}{8}$. Thus

$$G'(4) = 2^{g(4)}(\ln 2)g'(4) = 2^3(\ln 2)\frac{1}{8} = \ln 2.$$ 

Problem 6. (10pts) Compute one of the following two integrals. Indicate clearly which one you want graded.

(a–10pts) Compute \( \int \frac{1}{x \ln x \sqrt{(\ln x)^2 - 1}} \, dx \).

Solution. Let \( u = \ln x \), so that the lie is \( du = \frac{1}{x} \, dx \). Then
\[
\int \frac{1}{x \ln x \sqrt{(\ln x)^2 - 1}} \, dx = \int \frac{1}{u \sqrt{u^2 - 1}} \, du.
\]
Of course, \( \frac{d}{dx} \sec^{-1} x = \frac{1}{x \sqrt{x^2 - 1}} \), so
\[
\int \frac{1}{u \sqrt{u^2 - 1}} \, du = \sec^{-1} u + C = \sec^{-1}(\ln x) + C.
\]

(b–10pts) Compute \( \int \frac{1}{(1 - (\sin^{-1} x)^2)(\sqrt{1 - x^2})} \, dx \).

Solution. Let \( u = \sin^{-1} x \), so that the lie is \( du = \frac{1}{\sqrt{1 - x^2}} \, dx \). Then
\[
\int \frac{1}{(1 - (\sin^{-1} x)^2)(\sqrt{1 - x^2})} \, dx = \int \frac{1}{(1 - u^2)} \, du.
\]
Of course, \( \frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2} \), so
\[
\int \frac{1}{(1 - u^2)} \, du = \tanh^{-1} u + C = \tanh^{-1}(\sin^{-1} x) + C.
\]