Instructions: You may have 48 hours to work on this exam, and should note your start time and end time in the spaces below. You should not, of course, look at the exam before your start time, and during your chosen 48 hours, may consult only your notes, our text, and solutions that I have provided. After you have finished, please do not discuss the exam with anyone who is not finished as well. You should turn the exam at the next class period after your 48 hours has expired, Thursday May 31 at the latest.

Name: ________________________________

Start time: ________________

End time: ________________
Problem 1. Suppose that $f_n$ for $n \in \mathbb{N}$ and $f$ are Lebesgue integrable functions, that $f_n \to f$ almost everywhere, and that \( \{f_n(x)\}_{n=1}^{\infty} \) converges for all $x \in \mathbb{R}$. Prove that \( \int \lim_{n \to \infty} f_n \, dx = \int f \, dx \).

Problem 2. Let $E = [0,1]$, $\epsilon > 0$ be given, and $f_n = x^n \chi_E$ for $n \in \mathbb{N}$.

(1) Find a set $A$ such that $m(A) < \epsilon$ and $f_n \to 0$ uniformly on $A^c$ (your description of $A$ will naturally contain an $\epsilon$).

(2) Without using a dominated convergence theorem, prove that \( \lim_{n \to \infty} \int f_n \, dx = 0 \).

Problem 3. Let $f$ be a nonnegative, integrable function.

(1) Prove that \( \lim_{N \to \infty} \int_{-N}^{N} f(x) \, dx = \int_{\mathbb{R}} f(x) \, dx \).

(2) Let $g_N(x) = \min\{f(x), N\}$. Prove that \( \lim_{N \to \infty} \int_{\mathbb{R}} g_N(x) \, dx = \int_{\mathbb{R}} f(x) \, dx \).

Problem 4. Suppose that $f_n$ for $n \in \mathbb{N}$ and $f$ are positive integrable functions, $f_n \to f$ pointwise, and \( \lim_{n \to \infty} \int f_n(x) \, dx = 0 \). Prove that \( \int f \, dx = 0 \).

Problem 5. Prove the following form of the Lebesgue dominated convergence theorem: suppose $g$ and $f_n$ for $n \in \mathbb{N}$ are integrable and $|f_n| \leq g$ for all $n$. If $f_n$ converges almost everywhere to an integrable function $f$, then \( \lim_{n \to \infty} \int f_n \, dx = \int f \, dx \).

Problem 6. Suppose that $S$ is a bounded set in $[0,1]^2$ with strictly positive area. Prove that $S$ contains an open set.

Problem 7. Prove that \( \lim_{||N|| \to 0} a^+(N) = A^+(N) \).

Problem 8. Compute $A^-(S)$ and $A^+(S)$ for $S = \{(x,y) \in [0,1]^2 \mid x = y\}$.

Problem 9. A step function $S(x,y)$ on a rectangle $R$ is a bounded function for which there is a net $N$ on $R$ such that $S(x,y)$ is constant on $R_i$, that is $S(x,y)$ is constant on the interior of each subrectangle determined by $N$. Prove that any such step function is integrable on $R$ and that if $S(x,y) = k_i$ on $R_i$, then \( \int \int_{R} S(x,y) \, dA = \sum_{i=1}^{n} k_i A(R_i) \).