

**Calculus IV**  
**Math 241 Spring 2005**  
 Professor Ben Richert

**Exam 1**  
**Solutions**

**Problem 1.** (10 pts) Consider the surface given by the equation  $r^2 + z^2 = 8$ .

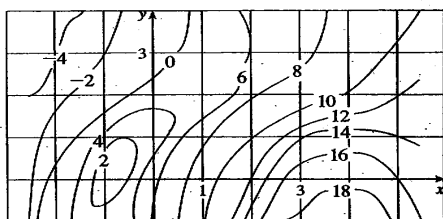
(a - 5 pts) Demonstrate that  $(r, \theta, z) = (2, \pi, 2)$  is a point on the surface.

*Solution.* We just need to show that  $(2, \pi, 2)$  satisfies the equation  $r^2 + z^2 = 8$ , but this is clear because  $r^2 + z^2 = (2)^2 + (2)^2$  does equal 8. □

(b - 5 pts) Is  $(x, y, z) = (1, \sqrt{3}, 2)$  a point on the surface? (Note that the point is given in rectangular coordinates).

*Solution.* We need to change  $(1, \sqrt{3}, 2)$  into cylindrical coordinates, and then proceed as in part (a). So if  $x = 1$ ,  $y = \sqrt{3}$  and  $z = 2$ , then  $r = \sqrt{x^2 + y^2} = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$ ,  $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$ , and  $z = 2$ . Now the point  $(2, \pi/3, 2)$  is on the surface because  $(2, \pi/3, 2)$  satisfies the equation  $r^2 + z^2 = 8$ , that is  $r^2 + z^2 = 4 + 4 = 8$ . □

**Problem 2.** (20 pts) A contour map is given for a function  $f(x, y)$ .



(a - 5 pts) Give an estimate for  $f(3, 2)$ .

*Solution.* Since the point  $(3, 2)$  sits half-way between the level curve  $f(x, y) = 8$  and the level curve  $f(x, y) = 10$ , we estimate that  $f(3, 2) \approx 9$ . □

(b - 5 pts) Give an estimate for  $f_x(3, 2)$ .

*Solution.* If we fix  $y = 2$  and let  $x$  vary, we see that  $f(2.2, 2) \approx 8$  and  $f(3.8, 2) \approx 10$ . So we estimate that  $f_x(3, 2) \approx \Delta z / \Delta x = \frac{f(3.8, 2) - f(2.2, 2)}{3.8 - 2.2} = \frac{10 - 8}{3.8 - 2.2} = 2/1.6 = 1.25$ . □

(c - 5 pts) Give an estimate for  $f_y(3, 2)$ .

*Solution.* If we fix  $x = 3$  and let  $y$  vary, we see that  $f(3, 1.6) \approx 10$  and  $f(3, 2.6) \approx 8$ . So we estimate that  $f_y(3, 2) \approx \Delta z / \Delta y = \frac{f(3, 2.6) - f(3, 1.6)}{2.6 - 1.6} = \frac{8 - 10}{1} = -2$ . □

(d - 5 pts) Use (a), (b), and (c) to give an approximate equation for the tangent plane to  $f(x, y)$  at  $(3, 2, f(3, 2))$ .

*Solution.* We know an equation for the tangent plane to the surface  $f(x, y)$  at the point  $(x_0, y_0, f(x_0, y_0))$ : it is  $z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ . In our situation, we can use our estimates for  $f(3, 2)$ ,  $f_x(3, 2)$ , and  $f_y(3, 2)$  to estimate that the tangent plane is  $z - 9 = 1.25(x - 3) - 2(y - 2)$ . □

**Problem 3.** (10 pts) Where is the function

$$f(x, y) = \begin{cases} \frac{x + y + 27}{2x^2 + y^2 + 1} & \text{for } \{x, y\} \neq \{0, 0\} \\ 2 & \text{for } \{x, y\} = \{0, 0\} \end{cases}$$

continuous?

*Solution.* We know that a rational function is continuous everywhere on its domain, that is, everywhere for which the denominator is not zero. Since the denominator of  $\frac{x+y+27}{2x^2+y^2+1}$  is  $2x^2+y^2+1$ , and  $2x^2, y^2 \geq 0$  for all  $x, y$ , it must be that  $2x^2+y^2+1$  is always at least 1 and hence never zero. Thus  $\frac{x+y+27}{2x^2+y^2+1}$  is continuous for all  $\{x, y\} \neq \{0, 0\}$ . We conclude that  $f(x, y)$  is continuous at least for all  $\{x, y\} \neq \{0, 0\}$ . To decide whether  $f(x, y)$  is continuous at  $\{0, 0\}$  we need to decide whether or not  $\lim_{\{x, y\} \rightarrow \{0, 0\}} f(x, y) = 2$ . Now

$$\lim_{\{x, y\} \rightarrow \{0, 0\}} f(x, y) = \lim_{\{x, y\} \rightarrow \{0, 0\}} \frac{x+y+27}{2x^2+y^2+1} = \lim_{\substack{\{x, y\} \rightarrow \{0, 0\} \\ \text{along } y=0}} \frac{x+y+27}{2x^2+y^2+1} = \lim_{x \rightarrow 0} \frac{x+27}{2x^2+1} = 27$$

(the last equality because  $\frac{x+27}{2x^2+1}$  is a rational function and hence is continuous everywhere on its domains). But  $27 \neq 2$ , so we conclude that  $f(x, y)$  is not continuous at the origin.  $\square$

**Problem 4.** (10 pts) Suppose that  $x, y, z$  are related by the equation  $yx = \ln(x+z)$ . What is the partial derivative of  $z$  with respect to  $x$  at the point  $(e/2, 2/e, e/2)$ ?

*Solution.* We use the fact that if  $F(x, y, z) = k$  for  $k$  a constant, then  $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ . Let  $F(x, y, z) = \ln(x+z) - yx$ . Then

$$\begin{aligned} F_x(x, y, z) &= \frac{1}{x+z} - y \\ F_z(x, y, z) &= \frac{1}{x+z}, \end{aligned}$$

so

$$\begin{aligned} F_x(e/2, 2/e, e/2) &= \frac{1}{e/2 + e/2} - 2/e = 1/e - 2/e = -1/e \\ F_z(e/2, 2/e, e/2) &= \frac{1}{e/2 + e/2} = 1/e. \end{aligned}$$

We conclude that

$$\frac{\partial z}{\partial x}(e/2, 2/e, e/2) = -\frac{-1/e}{1/e} = 1. \quad \square$$

**Problem 5.** (20 pts) Suppose that  $f(x, y) = x^2y - 6x^2 - 2y^2 - y + 100$  is a function which describes the concentration of algae in a lake  $x$  meters east and  $y$  meters north of a certain buoy. You are sitting in a boat at the point  $(1, 2)$ .

(a - 5 pts) In what direction is the concentration of algae increasing fastest (from where you sit)?

*Solution.* According to a theorem, the direction of max increase is  $\nabla f(1, 2)$ . We compute that  $\nabla f = \langle 2xy - 12x, x^2 - 4y - 1 \rangle$ , so the direction of max increase is  $\nabla f(1, 2) = \langle 2(1)(2) - 12, (1)^2 - 4(2) - 1 \rangle = \langle -8, -8 \rangle$ . That is southwest.  $\square$

(b - 5 pts) What is the rate of change of the concentration of algae in the northwest direction (again, from where you sit)?

*Solution.* A vector pointing directly northwest is  $\langle -1, 1 \rangle$ . We can turn this into a unit vector by dividing by its length, which is  $\sqrt{(-1)^2 + 1^2} = \sqrt{2}$ . So we are asked to compute the directional derivative of  $f(x, y)$  at  $(1, 2)$  in the direction  $\vec{u} = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ . The formula for  $D_{\vec{u}}f(1, 2)$  is  $\nabla f(1, 2) \cdot \vec{u}$ . We figured out above that  $\nabla f(1, 2) = \langle -8, -8 \rangle$ . So

$$D_{\vec{u}}f(1, 2) = \langle -8, -8 \rangle \cdot \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{8}{\sqrt{2}} + \frac{-8}{\sqrt{2}} = 0. \quad \square$$

(c - 10 pts) What are the coordinates of the point in the lake for which the concentration of the algae is highest? (You may assume the absolute maximum occurs at a local maximum point). (Use the back of the page if necessary).

*Solution.* To find the local maxima, we first set  $f_x = 0 = f_y$  and solve for  $x$  and  $y$ . In part (a) we computed that  $f_x = 2xy - 12x$  while  $f_y = x^2 - 4y - 1$ . Now if  $2xy - 12x = 2x(y - 6) = 0$ , then either  $x = 0$  or  $y = 6$ . This gives two cases that we must check.

If  $x = 0$ , then  $x^2 - 4y - 1 = 0$  implies that  $-4y - 1 = 0$ , or  $y = -1/4$ . So one critical point is  $(0, -1/4)$ .

If  $y = 6$ , then  $x^2 - 4y - 1 = 0$  implies that  $x^2 - 24 - 1 = 0$ , or  $x = \pm 5$ , so two other critical points are  $(\pm 5, 6)$ .

According to the text, we have a maximum point when both  $f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 > 0$  and  $f_{xx}(x, y) < 0$ , and neither a max nor a min if  $f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 < 0$ . So we compute that

$$\begin{aligned}f_{xx}(x, y) &= 2y - 12 \\f_{yy}(x, y) &= -4 \\f_{xy}(x, y) &= 2x.\end{aligned}$$

Then

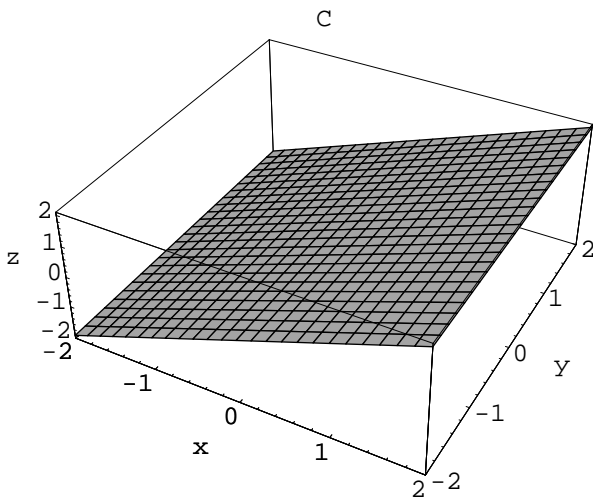
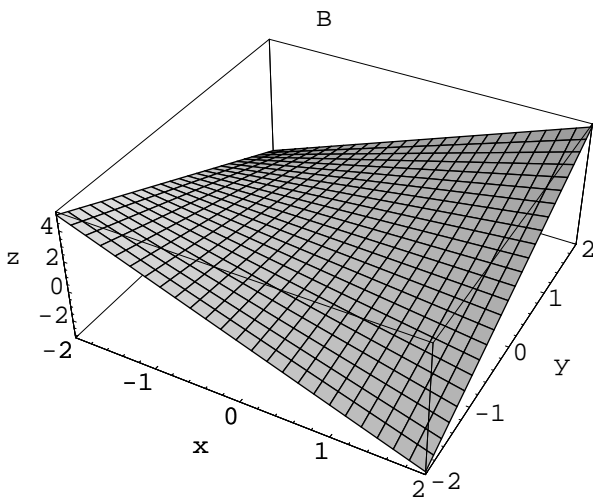
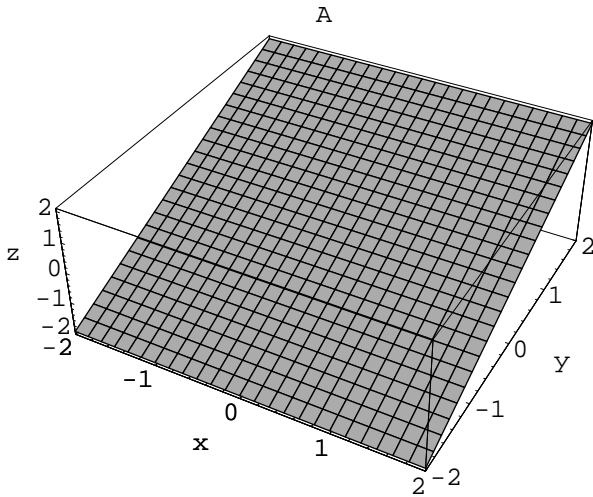
$$\begin{aligned}f_{xx}(0, -1/4)f_{yy}(0, -1/4) - (f_{xy}(0, -1/4))^2 &= (2(-1/4) - 12)(-4) - (0)^2 = 50 > 0 \text{ and} \\f_{xx}(0, -1/4) &= -25/2 < 0\end{aligned}$$

so  $(0, -1/4)$  is a maximum point. Also

$$\begin{aligned}f_{xx}(5, 6)f_{yy}(5, 6) - (f_{xy}(5, 6))^2 &= (2(6) - 12)(-4) - (10)^2 = -100 < 0 \\f_{xx}(-5, 6)f_{yy}(-5, 6) - (f_{xy}(-5, 6))^2 &= (2(6) - 12)(-4) - (-10)^2 = -100 < 0,\end{aligned}$$

so we conclude that neither  $(5, 6)$  or  $(-5, 6)$  is a max or a min. It must be, therefore, that the maximum concentration occurs at the point  $(0, -1/4)$ .  $\square$

**Problem 6.** (10 pts) One of the following graphs corresponds to  $f(x, y)$ , one to  $f_x(x, y)$  and one to  $f_y(x, y)$ . Which is which? (Be sure to explain carefully ... remember, I don't actually care which is which!)



*Solution.* If we cut A by a plane corresponding to a constant  $y = k$ , then we get a curve with slope zero. So if A corresponds to  $f$ , then  $f_x$  is always zero. Of course, neither B or C behaves this way (neither is just the zero map), so we conclude that A does not correspond to  $f(x, y)$ . The same argument works to show that C cannot correspond to  $f$  (if we cut C by a plane corresponding to a constant  $x = k$ , then we get a curve with slope zero—so if C corresponds to  $f$ , then  $f_y$  is zero, but neither A nor B exhibit this behavior). We conclude that C does not correspond to  $f$ . So it must be that B is the graph of  $f(x, y)$ . Now note that cutting B by a plane corresponding to  $x = k$  for a constant  $k$  gives a line (and hence the slope of the curve one gets is constant). So we expect  $f_y(x, y)$  to be a function which is constant for each fixed  $x$  value. We see that C exhibits this behavior (if you fix an  $x$  value, the curve you obtain is simply a constant), but A does not. The same argument (after switching  $x$  and  $y$ ) works to show that A corresponds to  $f_x(x, y)$ . So  $A \leftrightarrow f_x(x, y)$ ,  $B \leftrightarrow f(x, y)$ , and  $C \leftrightarrow f_y(x, y)$ .  $\square$