Recall the following definition from the text:

**Definition 1.** Let $X$ be a nonempty set. Then $\langle X \rangle = \{x_1^{k_1} \cdots x_m^{k_m} \mid x_i \in X, k_i \in \mathbb{Z}, m \geq 1\}$. We want to prove that if $X$ is a nonempty set, $G = \langle X \rangle$, and $H$ is a finite subgroup of $G$ such that $xHx^{-1} \subseteq H$ for all $x \in X$, then $H \triangleleft G$.

Before we begin the proof, it will help to establish a few things. First, we need a way to reliably work through the elements of $G$. Now since $G = \langle X \rangle$, we know that for each element $g \in G$ there exists an $m \in \mathbb{N}$, elements $x_1, \ldots, x_m \in X$, and integers $k_1, \ldots, k_m \in \mathbb{Z}$ such that $g = x_1^{k_1} \cdots x_m^{k_m}$. Unfortunately, these expressions are not unique. To get around this, note that we do not need the exponent zero to generate $xHx$. Let $G$ be a nonempty subset of $\langle X \rangle$, and integers $1 \leq k_1, \ldots, k_m \in \mathbb{Z} - \{0\}$. Thus, for each $g \in G - \{1\}$, the set

$$\left\{|k_1| + \cdots + |k_m| \mid x_1^{k_1} \cdots x_m^{k_m} = g \text{ for some } m \in \mathbb{N}, x_1, \ldots, x_m \in X, k_1, \ldots, k_m \in \mathbb{Z} - \{0\}\right\} \subseteq \mathbb{N}$$

is nonempty and hence contains a minimal element by the Well Ordering Axiom. Thus we can make the following definition.

**Definition 2.** Let $X$ be a nonempty subset of a group $G$. For each $g \in \langle X \rangle - \{1\}$ we define the absolute degree of $g$ to be the minimal element of the set

$$\left\{|k_1| + \cdots + |k_m| \mid x_1^{k_1} \cdots x_m^{k_m} = g \text{ for some } m \in \mathbb{N}, x_1, \ldots, x_m \in X, k_1, \ldots, k_m \in \mathbb{Z} - \{0\}\right\}.$$

We take the absolute degree of the multiplicative identity to be zero.

Now we can decompose $G$ as the disjoint union of elements of different absolute degrees, that is, writing $G_i$ to be the elements of $G$ of absolute degree $i$, we have $G = \bigcup_{i=0}^{\infty} G_i$. Thus to prove that $gHg^{-1} \subseteq H$ for all $g \in G$ it is enough to prove that, for all $n \in \mathbb{N}$, if $g$ is absolute degree $n$, then $gHg^{-1} \subseteq H$. We will, of course, prove the latter statement by induction.

Before we proceed we need one more fact.

**Lemma 1.** Let $G$ be a group and $X$ be a nonempty subset of $G$. If $H < G$ is finite and such that $xHx^{-1} \subseteq H$ for all $x \in X$, then $x^{-1}Hx \subseteq H$ for all $x \in X$.

**Proof.** Let $x \in X$. Since $|xHx^{-1}| = |H|$ (as we have shown before), it follows that $xHx^{-1} = H$. Now if $h \in H$, then there is $h_1 \in H$ such that $xh_1x^{-1} = h$ (since $xHx^{-1} = H$), whence $x^{-1}hx = h_1 \in H$. Of course $h$ was arbitrary, so we conclude that $x^{-1}Hx \subseteq H$ as required. \qed

Now we prove that main theorem in question.
**Theorem 1.** If $X$ is a nonempty set, $G = \langle X \rangle$, and $H \leq G$ is such that $xHx^{-1} \subseteq H$ for all $x \in X$, then $H \triangleleft G$.

**Proof.** We know that to show $H$ is normal, it is enough to show that $gHg^{-1} \subseteq H$ for all $g \in G$. As above, let $G_i$ to be the elements of $G$ of absolute degree $i$. Then since $G = \cup_{i=0}^\infty G_i$, it is enough to prove that $gHg^{-1} \subseteq H$ for all $g \in G_n$ for all $n \in \mathbb{N}$. We proceed by induction on $n$.

If $n = 0$, then $g = 1$, and obviously $1H1^{-1} \subseteq H$ as required. If $n = 1$, then $g \in G_1$ implies that $g = x$ or $g = x^{-1}$ for some $x \in X$. But we know that $xHx^{-1} \subseteq H$ by hypothesis and $x^{-1}Hx \subseteq H$ by Lemma (1), so the $n = 1$ case is proved.

Now suppose that $n > 1$ and $g \in G_n$. Thus there exists $m \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $k_1, \ldots, k_m \in \mathbb{Z} - \{0\}$ such that $g = x_1^{k_1} \cdots x_n^{k_n}$. If $h \in H$ is arbitrary we must show that $ghg^{-1} \in H$. Of course $x_mhx_m^{-1} \subseteq H$ by hypothesis, so there is $h_1 \in H$ such that $x_mhx_m^{-1} = h_1$. Note also that writing $g_1 = x_1^{k_1} \cdots x_n^{k_n}$, the absolute degree of $g_1$ is $n - 1$. Finally

$$g^{-1} = (x_1^{k_1} \cdots x_n^{k_n})^{-1} = x_m^{-k_m} \cdots x_1^{-k_1} = x_m^{-1}(x_m^{-k_m+1} \cdots x_1^{-k_1}) = x_m^{-1}(x_1^{k_1} \cdots x_n^{k_n})^{-1} = x_m^{-1}g_1^{-1}.$$  

Thus

$$ghg^{-1} = g_1x_mhx_m^{-1}g_1^{-1} = g_1h_1g_1^{-1}$$

while $g_1h_1g_1^{-1} \in H$ by the induction hypothesis. Since $h$ was arbitrary, $gHg^{-1} \subseteq H$, and since $g$ was arbitrary in $G_n$, $gHg^{-1} \subseteq H$ for all $g \in G_n$. By the principle of mathematical induction we conclude that $gHg^{-1} \subseteq H$ for all $g \in G_n$ for all $n \in \mathbb{N}_{\geq 0}$ as required. □

This is probably much more pedantic than Nicholson intended. He probably wanted us to prove the lemma (lemma (1)) and then cap off the whole argument by saying:

$$ghg^{-1} = (x_1^{k_1} \cdots x_n^{k_n})h(x_1^{k_1} \cdots x_n^{k_n})^{-1}(x_1^{k_1} \cdots x_n^{k_n})hx_m^{-1}(x_1^{k_1} \cdots x_n^{k_n})^{-1}$$

$$= (x_1^{k_1} \cdots x_n^{k_n})h(x_1^{k_1} \cdots x_n^{k_n})^{-1},$$

and then saying the magic induction hiding words “continuing similarly, we eventually arrive at an element in $H$.”

Doing that is certainly enough to get the idea of the proof across, but isn’t very careful. It hides the fact that we are doing induction as well as making it quite vague exactly what we are inducting over.

You may note that the theorem we just proved is not quite the same as problem number 2.8.13 in the text (not assigned). As far as I can tell problem 13a as stated in the book isn’t true. I don’t have a counter-example, but decided, having spent quite a bit of time thinking about it, just to prove the modified statement above so I could send this along.