Math 248, Methods of Proof, Fall 2013
Axioms of the Real Numbers

The Real Numbers consists of a set \( \mathbb{R} \) along with two functions \( + : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and a relation \( \leq \) on \( \mathbb{R} \) such that:

(A1) for all \( a, b, c \in \mathbb{R} \), \( +(+(a, b), c) = +(a, +(b, c)) \), or \( (a + b) + c = a + (b + c) \)

(A2) for all \( a, b \in \mathbb{R} \), \( a + b = b + a \)

(A3) there is an element \( 0 \in \mathbb{R} \) such that \( 0 + b = b \) for all \( b \in \mathbb{R} \); we call this element an additive identity (or a +-identity).

(A4) for each \( a \in \mathbb{R} \), there is a \( b \in \mathbb{R} \) such that \( a + b = 0 \); we refer to to this element as an additive inverse of \( a \) (or a +-inverse of \( a \)).

(A5) for all \( a, b, c \in \mathbb{R} \), \( \cdot(\cdot(a, b), c) = \cdot(a, \cdot(b, c)) \), or \( (a \cdot b) \cdot c = a \cdot (b \cdot c) \), or \( (ab)c = a(bc) \)

(A6) for all \( a, b \in \mathbb{R} \), \( ab = ba \)

(A7) there is \( 1 \in \mathbb{R} \) such that \( 1 \cdot b = b \) for all \( b \in \mathbb{R} \); we call this element a multiplicative inverse of \( b \) (or a \(-\)-inverse of \( b \))

(A8) for all \( a \in \mathbb{R} \) such that \( a \neq 0 \), there is \( b \in \mathbb{R} \) such that \( ab = 1 \)

(A9) for all \( a, b, c \in \mathbb{R} \), \( a(b + c) = ab + ac \)

(A10) \( 1 \neq 0 \)

(A11) Some more axioms regarding \( \leq \).