Problem 1 (20) Definitions and theorems (precision counts).

(a–4pts) Complete the following definition: Given sets $A$ and $B$, we say $A \subseteq B$ if . . .

Solution. $(\forall x \in A)(x \in B)$ or $(\forall x)(x \in A \rightarrow x \in B)$. 

(b–4pts) Complete the following definition: Given sets $A$ and $B$, then $A - B$ is . . .

Solution. the set $\{x \in A \mid x \notin B\}$

(c–4pts) State the Well Ordering Axiom.

Solution. If $S$ is a nonempty subset of the natural numbers then there is a minimal element in $S$, that is, there is $t \in S$ such that $t \leq t'$ for all $t' \in S$.

(d–4pts) State the theorem referred to as The Division Algorithm.

Solution. Given $a, b \in \mathbb{Z}$ such that $b > 0$ there exists unique $q, r \in \mathbb{Z}$ such that $a = bq + r$ and $0 \leq r < b$.

(e–4pts) Complete the following definition: Given $a, b \in \mathbb{N}$ not both zero, then we call $d$ the greatest common divisor of $a$ and $b$ if . . .

Solution. $d | a, d | b$, and if $d' \in \mathbb{N}$ such that $d' | a$ and $d' | b$, then $d' | d$.

Problem 2 (18pts) Quick computations/translations.

(a–3pts) State the converse of the following proposition: If Douglas dislikes dieting, then Bill barbecues beef brisket.

Solution. If Bill barbecues beef brisket then Douglas dislikes dieting.

(b–3pts) State the inverse of the following proposition: If I’ve told you once, I’ve told you a thousand times.

Solution. If I haven’t told you once, I haven’t told you a thousand times.

(c–3pts) Let $\mathbb{U}$ be the set of all people in the world and $\mathbb{V}$ be the set of all horses. Let $P(x, y)$ be the open sentence: Person $x$ likes horse $y$. Let $Q(x)$ be the open sentence: person $x$ does not prefer brown horses. Let $R(y)$ be the open sentence: horse $y$ is brown. Translate the following sentence into English: $(\forall x \in \mathbb{U})(\exists y \in \mathbb{V})(P(x, y) \land Q(x) \land R(y))$.

Solution. For every person $x$ in the world, there is a horse (among all the horses) such that person $x$ likes horse $y$ and person $x$ does not prefer brown horses and horse $y$ is brown.

Maybe a little smoother: Every person doesn’t prefer brown horses but likes some particular brown horse.
(d–3pts) Circle the statements that are logically equivalent to the negation of the proposition “all students love all math courses.”

(i) No student loves all math courses.
(ii) There is a student who does not love all math courses.
(iii) There is a student who loves a math course.

(e–3pts) Circle the statements that are logically equivalent to the negation of the proposition “all students love all math courses.”

(i) There is a student who loves all math courses.
(ii) Some students love all math courses.
(iii) There is a student who does not love one of the math courses.

(f–3pts) For each pair, circle the stronger proposition (or both if they are equivalent, or leave blank if neither implies the other):

(i) • [if George is tired or bored he will go to bed]
   • if George is tired he will go to bed

(ii) • $x \in A$
    • if $x \in A$ then $x \in B$

(iii) • [it is not the case that every moment of the day, if I am drinking coffee then I am not sleepy]
     • [there is a moment of the day during which I am drinking coffee and I am sleepy]

Problem 3 (10pts) Prove the following theorem: For all $a, b, c \in \mathbb{N}$, if $a \mid b$ and $a \mid c$, then $a^2 \mid bc$.

Solution. Since $a \mid b$ there is $t \in \mathbb{Z}$ such that $at = b$ (by definition). Similarly, since $a \mid c$ there is $t' \in \mathbb{Z}$ such that $at' = c$. Thus $bc = atat' = a^2(tt')$, that is, $a^2$ divides $bc$ as required.

Problem 4 (10pts) Let $A$, $B$, $C$, and $D$ be sets such that $A \subseteq C$ and $B \subseteq D$. Prove that $A \cap B \subseteq C \cap D$.

Solution. We need to show that for all $x$, if $x \in A \cap B$, then $x \in C \cap D$. So let $x \in A \cap B$ be arbitrary (as usual when proving conditional sentences, we begin by assuming the antecedent). Then $x \in A$ and $x \in B$ by definition. It follows that $x \in C$, since $A \subseteq C$ and $x \in D$ since $B \subseteq D$. Thus $x \in C \cap D$ by definition. Since $x$ was arbitrary, we conclude $A \cap B \subseteq C \cap D$ as required.

Problem 5 (10pts) Prove that the additive identity of $\mathbb{R}$ is unique. (You may only assume the real axioms for addition; that addition on $\mathbb{R}$ is well defined, closed, associative, and commutative, that an additive identity exists, and that each element has an additive inverse).

Solution. Suppose that $0, 0'$ are two additive inverses in $\mathbb{R}$. So $0 + a = a$ for all $a \in \mathbb{R}$, and in particular $0 + 0' = 0'$. Similarly, $a + 0' = a$ for all $a \in \mathbb{R}$ and hence $0 + 0' = 0$. We conclude that $0 = 0 + 0' = 0'$ or that $0 = 0'$. We conclude that the additive inverse of $\mathbb{R}$ is unique as required.
Problem 6 (10pts) Suppose that $x$ and $y$ are real numbers such that $x \geq 0 > y$. Prove that $|xy| = |x||y|$. You may use the normal facts about real addition, multiplication, and inequalities. Regarding absolute values, however, you should stick to the definition.

Solution. Recall the definition of $|x|$: 

$$|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}.$$ 

So in the case that $x = 0$, we have $|xy| = |0y| = |0| = 0 = |y| = |x||y|$ as required. If $x > 0$ and $y < 0$ then $xy < 0$ (by the usual rules about multiplying positive and negative values together). So by definition $|xy| = -xy$, $|x| = x$, and $|y| = -y$, and thus $|xy| = -xy = x(-y) = |x||y|$ as required.