Problem 1. (10pts) Solve the initial value problem: \( \frac{dy}{dx} = \frac{\cos(x)}{y^2}, \ y(2\pi) = 2. \)

Solution. This is a separable differential equation, so we cross multiply and integrate, yielding
\[
\int y^2 \, dy = \int \cos(x) \, dx,
\]
that is \( y^3/3 = \sin(x) + C. \) Solving for \( y \) gives
\[
y = \sqrt[3]{3\sin(x) + 3C}.
\]
Plugging in the initial value gives
\[
2 = y(2\pi) = \sqrt[3]{3\sin(2\pi) + 3C},
\]
or \(2 = \sqrt[3]{3C}, \) so \(3C = 8. \) Thus the solution to the initial value problem is
\[
y = \sqrt[3]{3\sin(x) + 8}.
\]
□

Problem 2 (10pts). Solve the differential equation: \( y' + 2xy = 2x. \)

Solution. This is a first order linear differential equation, and our formula states that the solution to \( y' + f(x)y = q(x) \) is
\[
y = e^{-\int F} \int e^{F} q(x) \, dx \text{ where } F'(x) = f(x).
\]
In this instance, \( f(x) = 2x \) so we take \( F(x) = x^2, \) whence
\[
y = e^{-x^2} \int e^{x^2} 2x \, dx = \frac{1}{e^{x^2}} \left( e^{x^2} + C \right) = 1 + C e^{x^2}
\]
is the general solution. Note that the integral \( \int 2xe^{x^2} \, dx \) is computed via a simple \( u \)-substitution. □

Problem 3. (10pts) A certain stream deep in the Rocky mountains contains 1 g/L of dissolved chalk (that precious national resource). The stream flows into a vat at a rate of 4 L/m and mixes instantaneously. Water flows out of the vat from the bottom at a rate of 2 L/m. Initially, the vat contained 2 L of water. What is the concentration in the vat after 4 minutes?

Solution. Let \( V(t) \) be the volume of water in the tank at time \( t, \) that is \( V(t) = 2 + 2t \) (initially the tank contains 2 L, and net rate of inflow is the constant 2). Let \( A(t) \) be the amount of salt in the tank at time \( t, \) (whence \( A(0) = 0). \) We know that for a short time interval \( \Delta t, \)
\[
\Delta A = (\text{amount of salt in}) - (\text{amount of salt out})
\]
\[
\approx (\text{rate water flows in})(\text{concentration of salt in the water flowing in})(\text{time})
\]
\[
-(\text{rate water flows out})(\text{concentration of salt in the water flowing out})(\text{time})
\]
\[
\approx (4)(1)\Delta t - (2)(\text{concentration of salt in the water flowing out})\Delta t \approx 4\Delta t - 2\frac{A(t)}{2 + 2t} \Delta t,
\]
and we conclude that
\[
\frac{\Delta A}{\Delta t} \approx 4 - \frac{A}{1 + t}.
\]
Of course,
\[
\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}.
\]
The augmented matrix for this system is:
\[
\begin{pmatrix}
2 & 3 & 3 & 4 \\
2 & 2 & 3 & 3 \\
4 & 6 & 7 & 3 \\
3 & 6 & 6 & 3 \\
\end{pmatrix}
\]

Solution. The augmented matrix for this system is:
\[
\begin{pmatrix}
1 & 2 & 2 & 3 & 4 \\
2 & 2 & 3 & 3 & 2 \\
4 & 6 & 7 & 3 & 1 \\
3 & 6 & 6 & 3 & 3 \\
\end{pmatrix}
\]

so row reducing in the usual way, we obtain the equivalent matrix in reduced-row-echelon form
\[
\begin{pmatrix}
1 & 2 & 2 & 3 & 4 \\
0 & -2 & -1 & -3 & -6 \\
0 & 0 & 0 & -6 & -9 \\
0 & 0 & 0 & 6 & 9 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 2 & 3 & 4 \\
0 & 2 & 1 & 3 & 6 \\
0 & 0 & -6 & -9 & -9 \\
0 & 0 & 0 & 6 & 9 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 0 & -1/2 \\
0 & 1 & 2/3 & 3/4 \\
0 & 0 & 1 & 3/2 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Note that \( z \) is a free variable. So we have \( w = 3/2, z = t, y = 3/4 - 1/2t \) and \( x = -2 - t \) for all \( t \in \mathbb{R} \). In vector form, the solutions is
\[
\begin{pmatrix}
-2 - t \\
3/4 - 1/2t \\
t \\
3/2 \\
\end{pmatrix}
= \begin{pmatrix}
-2 \\
3/4 \\
t \\
3/2 \\
\end{pmatrix} + \begin{pmatrix}
-1 \\
0 \\
1/2 \\
0 \\
\end{pmatrix}
\]

for all \( t \in \mathbb{R} \).

\[\square\]

Problem 4. (10pts) Solve the system of equations:
\[
\begin{align*}
x & + 2y + 2z + 3w = 4 \\
2x & + 2y + 3z + 3w = 2 \\
4x & + 6y + 7z + 3w = 1 \\
3x & + 6y + 6z + 3w = 3 \\
\end{align*}
\]

Solution. The augmented matrix for this system is:
\[
\begin{pmatrix}
1 & 2 & 2 & 3 & 4 \\
2 & 2 & 3 & 3 & 2 \\
4 & 6 & 7 & 3 & 1 \\
3 & 6 & 6 & 3 & 3 \\
\end{pmatrix}
\]

This is again a first order linear differential equation. Write \( \frac{dA}{dt} + A \frac{1}{1+t} = 4 \) and take \( F(t) = \ln|1+t| \). We can drop the absolute values since \( t \) is always positive. According to our method,
\[
y = e^{-\ln(1+t)} \int e^{\ln(1+t)} 4 \ dt = e^{\ln(1+t)} \int 4(1+t) \ dt
\]
\[
= \frac{1}{1+t} (2(1+t)^2 + c) = 2(1+t) + \frac{c}{1+t}.
\]

Since \( 0 = A(0) = 2(1) + \frac{c}{1} \) we conclude that \( c = -2 \). So
\[
A(t) = 2(1+t) - \frac{2}{1+t},
\]

and the concentration after 4 minutes is \( A(4)/V(4) = \left(2\left(\frac{2}{5}\right)\right)/(2 \cdot 2 \cdot 4) \) g/L.

\[\square\]

Problem 5. (30pts) Consider the matrix
\[
A = \begin{pmatrix}
1 & 0 & 2 \\
2 & 4 & 1 \\
1 & 3 & 1 \\
\end{pmatrix}
\]

and it appears that the error in our approximation (recall that we assumed that the concentration of the water in the tank stayed constant during the time interval \( \Delta t \)) goes to zero as \( \Delta t \to 0 \), so we say
\[
\frac{dA}{dt} = 4 - \frac{A}{1+t} \quad \text{with} \quad A(0) = 0.
\]

This is again a first order linear differential equation. Write \( \frac{dA}{dt} + A \frac{1}{1+t} = 4 \) and take \( F(t) = \ln|1+t| \). We can drop the absolute values since \( t \) is always positive. According to our method,
\[
y = e^{-\ln(1+t)} \int e^{\ln(1+t)} 4 \ dt = e^{\ln(1+t)} \int 4(1+t) \ dt
\]
\[
= \frac{1}{1+t} (2(1+t)^2 + c) = 2(1+t) + \frac{c}{1+t}.
\]

Since \( 0 = A(0) = 2(1) + \frac{c}{1} \) we conclude that \( c = -2 \). So
\[
A(t) = 2(1+t) - \frac{2}{1+t},
\]

and the concentration after 4 minutes is \( A(4)/V(4) = \left(2\left(\frac{2}{5}\right)\right)/(2 \cdot 2 \cdot 4) \) g/L.

\[\square\]
(a–10pts) Compute $A^T A$. No English required.

Solution. $A^T = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 2 & 1 & 1 \end{bmatrix}$, so

$A^T A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 & 1 \cdot 0 + 2 \cdot 4 + 1 \cdot 3 & 1 \cdot 2 + 2 \cdot 1 + 1 \cdot 1 \\ 0 \cdot 1 + 4 \cdot 2 + 3 \cdot 1 & 0 \cdot 0 + 4 \cdot 4 + 3 \cdot 3 & 0 \cdot 2 + 4 \cdot 1 + 3 \cdot 1 \\ 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 & 2 \cdot 0 + 1 \cdot 4 + 1 \cdot 3 & 2 \cdot 2 + 1 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 & 11 & 5 \\ 11 & 25 & 7 \\ 5 & 7 & 6 \end{bmatrix}$

$\square$

(b–10pts) Compute $\det(A)$ No English required.

Solution. We expand down the second column:

$\det(A) = -0 \cdot \det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + 4 \cdot \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - 3 \cdot \det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = 4(1 - 2) - 3(1 - 4) = 5.$

$\square$

(c–10pts) Now consider the matrix $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 1 & k & 1 \end{bmatrix}$. For what one value of $k$ is $B$ singular (that is, for what one value of $k$ does $B$ fail to be invertible)?

Solution. We know that $B$ is invertible if and only if $\det(B) \neq 0$, and thus $B$ is singular if and only if $\det(B) = 0$. So we are trying to find $k$ such that $\det(B) = 0$. We already did most of this calculation above. Expanding down the second column, we have that

$\det(B) = -0 \cdot \det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + 4 \cdot \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - k \cdot \det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = 4(1 - 2) - k(1 - 4) = -4 + 3k$

and $-4 + 3k = 0$ if and only if $k = 4/3$.

$\square$

Problem 6. (10pts) Suppose you have a matrix equation $A \vec{x} = \vec{0}$, for a $3 \times 3$ matrix $A$, and are certain that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a solution. Meanwhile, your roommate seems to have found another $3 \times 3$ matrix $C$ such that $CA = I_3 = AC$. Explain how you can tell that one of you must be making a mistake.

Solution. Recall the theorems which say 1) that $A \vec{x} = \vec{0}$ has a unique solution if and only if $\det A \neq 0$ (so if there are infinitely many solutions then $\det A = 0$), and 2) that $A$ is invertible if and only if $\det A \neq 0$.

Recall further that the homogeneous system $A \vec{x} = \vec{0}$ has either a unique solution (namely $\vec{x} = \vec{0}$) or infinitely many solutions. Thus if $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{0}$ we can conclude that this system has infinitely many solutions (having identified two such), and thus that $\det A = 0$.

If, however, there is a matrix $C$ such that $AC = I_3 = CA$, then $A$ is invertible ($C$ being the inverse of $A$), and hence $\det A \neq 0$.

Of course we can’t have $\det A = 0$ and $\det A \neq 0$ simultaneously, so either you or your roommate must be mistaken ... probably your roommate.
Extra Credit (2pts). Consider the differential equation \( y' = x^2 y \). Below are three graphs of its slope field. On one of these graphs I have also plotted an equilibrium solution. On another, I also plotted an isocline, and on another I plotted a non-constant solution. Explain which is which.

\[ \begin{align*} 
\text{Solution.} \quad \text{An equilibrium solution is a solution } y = c \text{ for } c \text{ a constant. The second version of the slope field below is the only one containing the graph of a constant function (i.e. a flat curve), so it must contain the equilibrium solution (in particular, the equilibrium solution } y = 0). \text{ The non constant solution must follow the slope field lines (the slopes are, after all, telling us the slope at each point of any solution going through that point), and that is clearly the case in the first slope field shown below. Finally, isoclines are curves of the form } y' = x^2 y = c, \text{ that is } y = c/x^2 \text{ for some constant } c. \text{ That fits the third picture given below. Note that all slope marks on the isocline should point in the same direction, but that is difficult to see on this graph since the computer didn't choose points exactly on the isocline when it rendered the slope field.} \end{align*} \]

\[ \begin{align*} 
\end{align*} \]