Problem 1 (30pts) Do each of the following.

(a–10pts) State the Well Ordering Axiom (the Well Ordering Proposition, in the book’s notation). You may not use the word minimal without definition.

\[ \text{Solution.} \] The Well Ordering Axiom states that any nonempty subset of \( \mathbb{N} \) contains a minimal element, that is, given \( S \subseteq \mathbb{N} \) such that \( S \neq \emptyset \), there is \( t \in S \) such that \( t \leq s \) for all \( s \in S \).

(b–10pts) Finish the following definition: Given a set \( A \) and a subset \( A \subseteq \mathcal{P}(A) \), we say \( A \) is a partition of \( A \) if . . .

\[ \text{Solution.} \] . . . the following conditions hold:
1. \( \emptyset \notin A \),
2. for all \( X, Y \in A \), either \( X = Y \) or \( X \cap Y = \emptyset \), and
3. \( \bigcup_{X \in A} X = A \).

(c–10pts) Finish the following definition: Given \( A \) and \( B \) sets, we say that a relation \( f \) from \( A \) to \( B \) is a function if . . .

\[ \text{Solution.} \] . . . for all \( a \in A \) there is a unique \( b \in B \) such that \( (a, b) \in f \).

Problem 2 (20pts) Let \( A = \{1, 2, 3, 4, 5, 6, 7, 8\} \), \( B = \{3, 4, 5, 6, 7\} \), and \( f : A \to B \) be the function \( f = \{(1, 3), (2, 3), (3, 5), (4, 6), (5, 5), (6, 4), (7, 6), (8, 3)\} \).

(a–5pts) Prove that \( f \) is injective or give a counter-example.

\[ \text{Solution.} \] A function \( f : A \to B \) is injective if and only if, for all \( x, y \in A \), if \( f(x) = f(y) \) then \( x = y \). This obviously fails here as \( f(1) = 3 = f(2) \), but \( 1 \neq 2 \).

(b–5pts) Compute \( f^{-1} \).

\[ \text{Solution.} \] This is simply the relation from \( B \) to \( A \) given by reversing all pairs of \( f \). So \( f^{-1} = \{(3, 1), (3, 2), (5, 3), (6, 4), (5, 5), (4, 6), (6, 7), (3, 8)\} \).

(c–5pts) Compute \( f^{-1}(\{5, 6\}) \).

\[ \text{Solution.} \] Recall that \( f^{-1}(Y) = \{a \in A \mid f(a) \in Y\} \). So here, \( f^{-1}(\{5, 6\}) = \{3, 4, 5, 7\} \).

(d–5pts) Let \( g = f\vert_{\{1, 2, 3\}} \). Compute \( g^{-1}(\{5, 6\}) \).
Solution. The key is to write down $g$. Recall that $f|_X = \{(a, b) \in f \mid a \in X\}$. So

$$g = f|_{\{1, 2, 3\}} = \{(1, 3), (2, 3), (3, 5)\}$$

Thus $g^{-1}(\{5, 6\}) = \{3\}$. \qed

**Problem 3** (15pts) Prove that $n^3 + 5n + 6$ is divisible by 3 for all $n \in \mathbb{N}$.

**Solution.** Let

$$S = \{k \in \mathbb{N} \mid k^3 + 5k + 6 \text{ is divisible by 3}\}.$$  

For the base case, we observe that $1 \in S$ because $1^3 + 5(1) + 6 = 12$ is divisible by 3. Now suppose that $n \in S$. Then $n^3 + 5n + 6$ is divisible by 3 by the induction hypothesis, say $n^3 + 5n + 6 = 3t$ for some $t \in \mathbb{N}$. Of course,

$$(n + 1)^3 + 5(n + 1) + 6 = n^3 + 3n^2 + 3n + 1 + 5n + 5 + 6$$

$$= n^3 + 5n + 6 + 3n^2 + 3n + 6 = 3t + 3(n^2 + n + 2) = 3(t + n^2 + n + 2),$$

which is obviously divisible by 3, so we conclude that $n + 1 \in S$. By the principle of mathematical induction, it follows that $S = \mathbb{N}$, that is for all $n \in \mathbb{N}$, $3 \mid (n^3 + 5n + 6)$ as required. \qed

**Problem 4** (15pts) Let $f : A \to B$ be a function and $X, Y \subseteq A$. Prove that $f(X \cap Y) \subseteq f(X) \cap f(Y)$.

**Solution.** Recall that by definition

$$f(X) = \{b \in B \mid (\exists a \in X)(f(a) = b)\},$$

and hence $a \in X \implies f(a) \in f(X)$. Now let $b \in f(X \cap Y)$ be arbitrary (we need to show that $b \in f(X) \cap f(Y)$). So there is $a \in X \cap Y$ such that $f(a) = b$, that is, there is $a \in A$ such that $a \in X$, $a \in Y$, and $f(a) = b$. We conclude that $b = f(a) \in f(X)$ and $b = f(a) \in f(Y)$, so that $b \in f(X) \cap f(Y)$ as required. \qed

**Problem 5** (15pts) Suppose that $A$, $B$, and $C$ are sets and $f : A \to B$ and $g : B \to C$ are functions. Prove that $g$ is surjective if $g \circ f$ is surjective.

**Solution.** Let $c \in C$ be arbitrary. We need to show that there is $b \in B$ such that $g(b) = c$. Since $g \circ f : A \to C$ is surjective, there is $a \in A$ such that $(g \circ f)(a) = c$. So let $b = f(a) \in B$. We observe that $g(b) = g(f(a)) = (g \circ f)(a) = c$ as required. \qed