Problem 1. (10pts) Suppose that you own an ice-cream shop and after $t$ hours ($0 \leq t \leq 6$) have sold $f(t) = 2^{t+\sin t}$ scoop of chocolate—scoop of vanilla ice cream cones. How fast are you selling scoop of chocolate—scoop of vanilla cones 2 hours after you open the shop?

Solution. The speed at which you are selling scoop of chocolate—scoop of vanilla cones after 2 hours (that is, the rate of change of the number of cones sold at $t = 2$) is given by evaluating the derivative of $f(t)$ at $t = 2$. So

$$f'(t) = \frac{d}{dx}(2^{t+\sin t}) = 2^{t+\sin t}(\ln 2)(1 + \cos t)$$

(using the chain rule), and $f'(2) = 2^{2+\sin 2}(\ln 2)(1 + \cos 2)$ cones per hour. □

Problem 2. (10pts) Suppose that $p = f(t) = e^{4t-\frac{3}{2}t^2}$ gives the pH in Santa Margarita lake $t$ days after the end of the last rain of the winter. Find a formula for $f^{-1}(p)$ and state in words what this formula computes.

Solution. First solve $p = e^{4t-\frac{3}{2}t^2}$ for $t$. The algebra gives us:

$$\ln(p) = \ln \left(e^{4t-\frac{3}{2}t^2}\right) = \frac{4t - 1}{2t + 2}$$

$$\Rightarrow \ln(p)(2t + 2) = 4t - 1$$

$$\Rightarrow 2t \ln(p) + 2 \ln(p) = 4t - 1$$

$$\Rightarrow 2t \ln(p) - 4t = -2 \ln(p) - 1$$

$$\Rightarrow t(2 \ln(p) - 4) = -2 \ln(p) - 1$$

$$\Rightarrow t = \frac{-2 \ln(p) - 1}{2 \ln(p) - 4}.$$ 

So a formula for the inverse is

$$f^{-1}(p) = \frac{-2 \ln(p) - 1}{2 \ln(p) - 4}.$$ 

(note that we didn’t switch $t$ and $p$ since the formula asked for $f^{-1}(p)$, but that is a very minor point). This function computes that number of days since the last rain for which the lake has a given pH. □

Problem 3. (10pts) Compute the volume obtained by rotating the region bounded by $y = 0$, $y = \frac{1}{(1-3x^2)^{3/2}}$, $x = 0$, and $x = 1/2$ about the $y$-axis.
**Solution.** The formula for volume using shells is \( \int_a^b 2\pi xf(x)\,dx \) (and shells seems the best in this instance). For us, this becomes

\[
\int_0^1 \frac{2\pi x}{(1 - 3x^2)^{3/2}} \, dx.
\]

Let \( u = 1 - 3x^2 \), so that \( du/dx = -6x \) and the lie is that \( du/(-3) = 2x \, dx \). Note that \( x = 0 \implies u = 1 \) and \( x = 1/2 \implies u = 1/4 \). Then the integral becomes

\[
\int_{1/4}^1 \pi u^{-3/2} \, du = -\frac{\pi}{3} \int_{1}^{1/4} u^{-3/2} \, du = -\frac{\pi}{3} (-2)u^{-1/2} \bigg|_{1/4}^{1/4} = \left( \frac{2\pi}{3\sqrt{1/4}} - \frac{2\pi}{3} \right) = \frac{4\pi}{3} - \frac{2\pi}{3} = \frac{2\pi}{3}.
\]

\( \square \)

**Problem 4.** (10pts) It is difficult to live in the dorms, the main problem being the noise. You finally get fed up, and decide to report the average decibel level to your senator. After much study, you decide that the decibel level in your dorm is given by the function \( D(t) = 20\cos t \sqrt{1 + \sin^2 t} \) where \( t \) is the number of hours after midnight (on any given night). Compute the average decibel level from in your dorm from \( t = 0 \) to \( t = 3/2 \) (i.e. from midnight to 1:30am).

**Solution.** The formula for the average value of a function \( f(x) \) on \([a,b]\) is \( \frac{1}{b-a} \int_a^b f(x) \, dx \). In this instance, we have

\[
\frac{1}{3/2 - 0} \int_0^{3/2} \frac{20\cos t}{\sqrt{1 + \sin^2 t}} \, dt = \frac{40}{3} \int_0^{\sin(3/2)} \frac{1}{\sqrt{1 + u^2}} \, du = \frac{40}{3} \sin^{-1}(u) \bigg|_0^{\sin(3/2)} = \frac{40}{3} \left( \sin^{-1}(\sin(3/2)) - \sin^{-1}(0) \right) = \frac{40}{3} \sin^{-1}(\sin(3/2)),
\]

where we used the \( u \)-substitution \( u = \sin t \) (so that \( du/dt = \cos t \), the lie is that \( du = \sin t \, dt \), \( t = 0 \implies u = \sin 0 = 0 \), and \( t = 3/2 \implies u = \sin(3/2) \)), and the fact that \( \frac{d}{du} \sin^{-1} u = \frac{1}{\sqrt{1 + u^2}} \).

\( \square \)

**Problem 5.** (15pts) The \( e^{i\pi} \) fraternity (the beloved math fraternity on campus) has a conical tank filled with Kool-Aid\(^\text{TM}\). At a heated homework session, someone drops a piece of chalk (a precious national resource) into the tank, and they have to pump all the Kool-Aid\(^\text{TM}\) out in order to retrieve it. The tank is 1 meter tall, and has radius 1 meter at the top, while the Kool-Aid\(^\text{TM}\) in the tank has density 1 kilogram per cubic meter. How much work does it take to pump all the Kool-Aid\(^\text{TM}\) out of the top of the tank? (You may ignore the volume of the chalk).

**Solution.** We begin the usual process:

- Split the contents of the tank into \( n \) slabs of thickness \( \Delta x = \frac{1}{n} \).
- Label the the sides of the slabs \( x_0, x_1, \ldots \) as so on (so that the bottom of the \( i \)-th slab is \( x_i = i\Delta x \) from the top of the tank).
- We can estimate the amount of work to pump the \( i \)-th slab to the top of the tank (making the usual assumptions about shape and mass ... take a look at the picture):
work \approx (\text{force})(\text{distance})
\approx (\text{mass})(\text{acceleration})(\text{distance})
\approx (\text{density})(\text{volume})(\text{acceleration})(\text{distance})
\approx (1)(\pi(1 - x_i)^2\Delta x)(9.8)(x_i).

• The total work is thus approximately
\[ \sum_{i=1}^{n} 9.8\pi x_i (1 - x_i)^2 \Delta x. \]

• Since it appears that our estimate improves as \( n \) increases, we believe that the total work is given by
\[ \lim_{n \to \infty} 9.8\pi x_i (1 - x_i)^2 \Delta x. \]

Of course, the limit can be computed using the integral
\[ \int_{0}^{1} 9.8\pi x(1 - x)^2 \, dx = \int_{0}^{1} 9.8\pi (x - 2x^2 + x^3) \, dx = 9.8\pi \left( \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right) \bigg|_{0}^{1} = 9.8\pi \left( \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) \]
\text{joules of work.} \]
Problem 6. (10 pts) Consider the following graph, then match each of the entries in the left column with one entry from the right column.

\begin{itemize}
  \item \textbf{b} \quad f_1(x)
  \item \textbf{g} \quad f_2(x)
  \item \textbf{a} \quad f_3(x)
  \item \textbf{e} \quad f_4(x)
  \item \textbf{c} \quad f_5(x)
  \item \textbf{h} \quad \text{Domain of } e^x
  \item \textbf{f} \quad \text{Range of } x^2
  \item \textbf{d} \quad \text{Range of } e^x
\end{itemize}

\begin{itemize}
  \item (a) Graph of \( x^2 \), for \( x \geq 0 \)
  \item (b) Graph of \( 3^x \)
  \item (c) Graph of inverse of \( \left( \frac{1}{2} \right)^x \)
  \item (d) Domain of \( \frac{1}{\sqrt{x}} \)
  \item (e) Graph of \( \left( \frac{1}{2} \right)^x \)
  \item (f) \([0, \infty)\)
  \item (g) Graph of \( e^x \)
  \item (h) Range of \( \ln x \)
\end{itemize}