Problem 1. (10pts) You always drink too much coffee on exam days, but it typically doesn’t bother you until later on, usually during your afternoon nap. After the last exam, you dreamt that a giant (spherical) drop of coffee was growing in the corner of your dorm room. The radius of this coffee bubble was increasing at a rate of \( \frac{2}{\pi} \) meters per second. How fast was the volume of the precious sphere increasing when it (the volume) was \( \frac{4\pi}{3} \) cubic meters? By the way, the volume of a sphere is \( \frac{4\pi r^3}{3} \).

Solution. The chain rule tells us that since \( V \) and \( r \) are functions of \( t \), then

\[
\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt}
\]

We easily compute that

\[
\frac{dV}{dr} = \frac{d}{dr} \left( \frac{4\pi}{3} r^3 \right) = 4\pi r^2,
\]

and note that

\[
r = \sqrt[3]{\frac{4\pi}{3} V},
\]

so that

\[
r = \sqrt[3]{\frac{3}{4\pi} \frac{4\pi}{3}} = \sqrt{1} = 1
\]

when \( V = \frac{4\pi}{3} \). Since we are given that \( \frac{dr}{dt} = \frac{2}{\pi} \), it follows that

\[
\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 2\pi \frac{r}{\pi} \frac{2}{\pi} = 4
\]

cubic meters per minute when \( V = \frac{4\pi}{3} \). \( \square \)

Problem 2. (10pts) Suppose that your calculator is on the fritz. It still adds, subtracts, multiplies, and divides just fine, but the square root button is sticking (you spilled Koolaid \( \text{TM} \) on it). You have some strange urge to estimate \( \sqrt{9.1} \). Write a short paragraph explaining how to use Newton’s method to do this. Hint: consider \( f(x) = x^2 - 9.1 \).

Solution. Let \( f(x) = x^2 - 9.1 \) and observe that \( \sqrt{9.1} \) is a root of \( f(x) = x^2 - 9.1 = 0 \). Thus we can approximate \( \sqrt{9.1} \) by using Newton’s method to approximate a root of \( f(x) = 0 \) near \( x = 3 \). The first step is to make an initial estimate of the root, and taking \( x_1 = 3 \) seems like a pretty good idea. We now iteratively compute approximations of the root according to the rule

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 9.1}{2x_n}.
\]

We we continue until \( |x_{n+1} - x_n| \) is small (in general, we continue until \( |x_{n+1} - x_n| < 10^{-n} \) which gives \( n - 1 \) decimal places of accuracy).

Although this wasn’t necessary for your answer, I’ll go ahead and make the approximation (I am using a calculator). With \( x_1 = 3 \), we get

\[
x_2 \approx 3.016666667
\]
\[
x_3 \approx 3.016620626,
\]
\[
x_4 \approx 3.016620626
\]

Note that \( |x_4 - x_3| \leq 0.000000009 \) which is pretty small (so \( \sqrt{9.1} \approx 3.016620626 \) with 7 decimal places of accuracy). In fact, if I ask the calculator for the value of \( \sqrt{9.1} \) I get \( \approx 3.016620626 \). \( \square \)

Problem 3. (10pts) Suppose that now your calculator has stopped working entirely (except for the Tetris game you loaded, which still goes like a charm). Use a linearization of the function \( f(x) = \sqrt{x} \) to estimate \( \sqrt{9.1} \).
Solution. The equation for the linearization \( L_a(x) \) is \( L_a(x) = f(a) + f'(a)(x - a) \), and we know that \( L_a(x) \approx f(x) \) for \( x \) near \( a \). Since 9.1 is near 9, we compute \( L_9(9.1) \) to approximate \( f(9.1) = \sqrt{9.1} \). Now \( f'(x) = \frac{1}{2 \sqrt{x}} \), so we have \( L_9(x) = \sqrt{9} + \frac{1}{2 \sqrt{9}}(x - 9) = 3 - \frac{1}{6}(x - 9) \), and \( L_9(9.1) = 3 + \frac{1}{6}(9.1 - 9) = 3 + \frac{1}{60} = \frac{181}{60} \). Just for the sake of comparing with your answer in the previous problem, note that the calculator says that \( \frac{181}{60} \approx 3.016666667 \). □

Problem 4. (10pts) Your senior project is to design the perfect coffee cup. It should be cylindrical, without a lid or handle, made of titanium, and hold \( 8 \pi \) cubic inches of coffee. Coffee cup grade titanium costs \$10 per square inch. What is the minimum cost of such a cup?

Solution. We want to minimize cost given volume. The surface area for the coffee cup is \( S = \pi r^2 + 2 \pi rh \) (the area of the base plus the area of the side) where \( r \) is the radius of the bottom of the cylinder and \( h \) is the height.

Since the cup will cost \$10 dollars per square inch, it follows that the cost is \( C = 10S = 10\pi r^2 + 20\pi rh \). This is the objective equation. The constraint is that the volume of the cup is \( 8\pi \); so \( 8\pi = V = \pi r^2 h \). Solving \( 8\pi = \pi r^2 h \) for \( h \) yields \( h = \frac{8\pi}{\pi r^2} = \frac{8}{r^2} \) and hence we can write \( C = 10\pi r^2 + 20\pi r \left( \frac{8}{r^2} \right) = 10\pi r^2 + \frac{160\pi}{r} \). The domain for this function is \( r > 0 \) (obviously we can’t have \( r = 0 \) as then our coffee cup has no volume).

Now it is simply a matter of minimizing \( C = 10\pi r^2 + \frac{160\pi}{r} \) for \( r > 0 \). Note that the derivative of \( C \) with respect to \( r \) is \( C' = 20\pi r - \frac{160\pi}{r^2} \), and setting this equal to zero (in order to find all critical points) yields:

\[
20\pi r - \frac{160\pi}{r^2} = 0
\]

so that

\[
1 - \frac{8}{r^3} = \frac{0}{20\pi r} = 0,
\]

or

\[
1 = \frac{8}{r^3}
\]

and hence

\[
r^3 = 8
\]

so that \( r = 2 \).

It is worth noting that \( C'' = 20\pi + \frac{320}{r^3} > 0 \) if \( r = 2 \), so that \( C \) is minimized at \( r = 2 \) (there are no endpoints to check and the derivative always exists, so the absolute minimum, if it exists, occurs at \( r = 2 \); since \( \lim_{r \to 0^+} 10\pi r^2 + \frac{160\pi}{r} = \infty \) and \( \lim_{r \to \infty} 10\pi r^2 + \frac{160\pi}{r} = \infty \), we know that an absolute minimum does in fact exist).

We now compute that \( h = \frac{8}{r^2} = 2 \) inches when \( r = 2 \), so that the minimum cost of the coffee cup is \( 10\pi (2)^2 + 20\pi (2)(2) = 120\pi \) dollars. □
Problem 5. (15pts) Consider the following graph, then match each of the entries in the left column with one entry from the right column.

<table>
<thead>
<tr>
<th>D</th>
<th>( \lim_{x \to \infty} f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>( \lim_{x \to -\infty} (f(x) - (x + 3)) )</td>
</tr>
<tr>
<td>F</td>
<td>( N &gt; 0 ) such that (</td>
</tr>
<tr>
<td>B</td>
<td>slope of the linearization of ( f(x) ) at ( x = 13/2 )</td>
</tr>
<tr>
<td>E</td>
<td>( x ) such that ( f(x) ) takes its absolute minimum</td>
</tr>
<tr>
<td>J</td>
<td>absolute maximum of ( f(x) )</td>
</tr>
<tr>
<td>A</td>
<td>( x ) such that ( f(x) ) takes its absolute maximum</td>
</tr>
<tr>
<td>G</td>
<td>( x ) such that ( f'(x) &gt; 0 ) and ( f''(x) = 0 )</td>
</tr>
<tr>
<td>N</td>
<td>( x ) such that ( f'(x) = 0 ) and ( f''(x) &gt; 0 )</td>
</tr>
<tr>
<td>M</td>
<td>( M ) such that (</td>
</tr>
<tr>
<td>L</td>
<td>slope of the tangent at the point ( c ) satisfying the conclusion of the Mean Value Theorem for the interval ( \left[ \frac{11}{2}, \frac{15}{2} \right] )</td>
</tr>
<tr>
<td>K</td>
<td>antiderivative of ( \sin(x^2) + 2x^2 \cos(x^2) )</td>
</tr>
<tr>
<td>C</td>
<td>antiderivative of ( -\sin(x) \sin(x^2) + 2x \cos(x) \cos(x^2) )</td>
</tr>
<tr>
<td>I</td>
<td>antiderivative of ( -\sin(x) + 2x \cos(x^2) )</td>
</tr>
</tbody>
</table>

(A) 3  
(B) -1  
(C) \( \cos(x) \sin(x^2) \)  
(D) 3/2  
(E) Does not exist  
(F) 8  
(G) 1  
(H) 0  
(I) \( \cos(x) + \sin(x^2) \)  
(J) 7  
(K) \( x \sin(x^2) \)  
(L) -3/2  
(M) 1/2  
(N) 15/2