Problem 1. (10 pts) Consider the function \( f(x) = \begin{cases} x + 1 & \text{if } x > 1 \\ 2/x & \text{if } x \leq 1. \end{cases} \) Where is \( f(x) \) continuous?

Solution. We know that polynomials are continuous on all of \( \mathbb{R} \) and that rational functions are continuous wherever their denominators are nonzero. Thus \( 1 + x \) is continuous on all \( \mathbb{R} \) and \( 1/x \) is continuous on \((-\infty, 0) \cup (0, \infty) \). It follows that \( f(x) \) is continuous on \((1, \infty)\) (since \( f(x) = 1 + x \) on \((1, \infty)\)) and continuous on \((-\infty, 0) \cup (0, 1)\), but discontinuous at \( x = 0 \) (since \( f(x) = 1/x \) on \((-\infty, 0) \cup (0, 1)\)). It remains to check for continuity at \( x = 1 \). By definition, \( f(x) \) is continuous at \( x = 1 \) if \( \lim_{x \to 1-} f(x) = f(1) = 2 \). We now that \( \lim_{x \to 1-} f(x) = 2 \) if and only if \( \lim_{x \to 1-} f(x) = f(2) = 2 \) and \( \lim_{x \to 1+} f(x) = 2 \) (because \( f(x) = 1/x \) for \( x > 1 \) and \( 1 + x \) is continuous at \( x = 1 \)), and thus \( \lim_{x \to 1} f(x) = f(1) = 2 \). We conclude that \( f(x) \) is continuous at \( 2 \). To summarize, we have that \( f(x) \) is continuous on \((-\infty, 0) \cup (0, \infty) \), but discontinuous at \( x = 0 \). \( \square \)

Problem 2. (10 pts) What is the equation of the tangent line to \( x^2 + 4xy + y^2 = 13 \) at the point \( (2,1) \)?

Solution. We use implicit differentiation to compute \( y' \). Taking the derivative of both sides with respect to \( x \) (and remembering that \( y \) is a function of \( x \)), we find that \( \frac{d}{dx}(x^2 + 4xy + y^2) = \frac{d}{dx}13 \), that is, \( 2x + 4y + 4yy' + 2yy' = 0 \). Note the use of the product and chain rules to differentiate \( 4xy \), and the use of the chain rule to differentiate \( y^2 \). Now we solve for \( y' \), so \( y'(4x + 2y) = -(2x + 4y) \), that is \( y' = \frac{-2x + 4y}{4x + 2y} \). At the point \( (2,1) \), \( y' = \frac{-2(2) + 4(1)}{4(2) + 2(1)} = \frac{-4}{10} = \frac{-2}{5} \). Using the point slope formula, we get the equation of the tangent line is thus \( y - 1 = \frac{-2}{5}(x - 2) \). \( \square \)

Problem 3. (20 pts) A certain chalk company, Nordic Enterprises, mines the raw materials for their chalk at a secret location on the Central Coast. You are privileged enough to visit the mine, and notice that the amount of chalk dust in the air \( t \) hours after the beginning of an eight hour shift is \( p(t) = \frac{24}{5} \sqrt{3 + t + \sqrt{t}} \) pounds, \( 0 \leq t \leq 8 \).

(a - 10 pts) Find an equation which computes the number of pounds of chalk dust entering the air per hour after \( t \) hours.

Solution. We know that the number of pounds of chalk dust entering the air per hour after \( t \) hours is the rate of change of the number of pounds of chalk dust in the air after \( t \) hours. Since the derivative measures rates of change, the equation we are looking for is \( p'(t) \). The chain rule is required to take this derivative. Recall that \( \frac{d}{dt}(f(g(t))) = f'(g(t))g'(t) \). In our situation we take \( f(t) = \frac{24}{5} t^{3/2} \) and \( g(t) = 3 + t + t^{1/2} \). The relevant table is:

\[
\begin{array}{c|c|c}
  f(t) &=& \frac{24}{5} t^{3/2} \\
  f'(t) &=& \frac{12}{5} t^{-1/2} = \frac{12}{5\sqrt{t}} \\
  g(t) &=& 3 + t + t^{1/2} \\
  g'(t) &=& 1 + \frac{1}{2} t^{-1/2} = 1 + \frac{1}{2\sqrt{t}} \\
\end{array}
\]

Professor Ben Richert
Calculus I
Math 141 Fall 2006
Exam 1
Solutions
Problem 5. (10pts) A certain friend of yours runs a small business. In fact, she sells sea shells, down by the seashore, and
Thus.

Problem 4. (10pts) Suppose that \( f(3) = \pi, \ f'(3) = 2, \ g(3) = 4, \ g'(3) = 1 \) and \( h(x) = \frac{f(x) - g(x)}{\cos(f(x))} \). What is \( h'(3) \)?

Solution. We use the quotient and chain rules. In general \( \frac{d}{dx} \frac{k(x)}{m(x)} = \frac{m(x)k'(x) - k(x)m'(x)}{(m(x))^2} \). In our situation \( k(x) = f(x) - g(x) \) and \( m(x) = \cos(f(x)) \), so that the relevant table is:

<table>
<thead>
<tr>
<th>( k(x) = f(x) - g(x) )</th>
<th>( m(x) = \cos(f(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k'(x) = f'(x) - g'(x) )</td>
<td>( m'(x) = -\sin(f(x))f'(x) )</td>
</tr>
</tbody>
</table>

The lower right entry in the table comes from differentiating \( \cos(f(x)) \) using the chain rule. In particular, set \( k(x) = \cos x \) and \( m(x) = f(x) \), then we get the table:

<table>
<thead>
<tr>
<th>( k(x) = \cos x )</th>
<th>( m(x) = f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k'(x) = -\sin x )</td>
<td>( m'(x) = f'(x) )</td>
</tr>
</tbody>
</table>

so that the derivative, \( \frac{d}{dx}(k(m(x))) = k'(m(x))m'(x) = -\sin(f(x))f'(x) \), as stated above.

Thus \( h'(x) = \frac{\cos(f(x))(f'(x) - g'(x)) - (f(x) - g(x))(-f'(x)\sin(f(x)))}{\cos^2(f(x))} \),

and so

\[
h'(3) = \frac{\cos(f(3))(f'(3) - g'(3)) - (f(3) - g(3))(-f'(3)\sin(f(3)))}{\cos^2(f(3))} \\
= \frac{\cos(\pi)(2 - 1) - (\pi - 4)(-2\sin(\pi))}{\cos^2(\pi)} = \frac{-1(1) - (\pi - 4)(-2(0))}{(-1)^2} = -1.
\]

Problem 5. (10pts) A certain friend of yours runs a small business. In fact, she sells sea shells, down by the seashore, and
hires you to do the books. You keep track of sales and discover that the number of shells she has sold after \( w \) weeks is:

<table>
<thead>
<tr>
<th>week ( w )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td># of shells sold</td>
<td>0</td>
<td>5</td>
<td>7</td>
<td>13</td>
<td>16</td>
<td>22</td>
<td>28</td>
</tr>
</tbody>
</table>

Use the table to estimate the rate at which shells are sold after 2 weeks.

Solution. We can estimate the instantaneous rate of change of sales (in number of shells per week), by computing the average rate of change of sales (in number of shells per week) for a small time interval containing \( w = 2 \). We use the interval from \( w = 1 \) to \( w = 3 \). The average rate of change over that time is \( \frac{13 - 7}{3 - 1} = \frac{6}{2} = 3 \) shells per week, and this estimates the derivative at \( w = 2 \).

Problem 6. (10pts) Consider the following graph, then match each of the entries in the left column with one entry from the right column.
\[
\begin{align*}
E & \lim_{x \to -0} f(x) \\
C & \lim_{x \to 1^+} f(x) \\
I & f(1) \\
G & f'(4) \\
B & x \text{ such that } f'(x) \text{ does not exist} \\
A & \delta > 0 \text{ such that if } |x - 2| < \delta, \text{ then } |f(x) - 7/2| < 1/2 \\
H & \delta > 0 \text{ such that if } |x - 3| < \delta, \text{ then } |f(x) - 2| < 1 \\
J & \text{interval on which } f(x) \text{ is continuous} \\
F & \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} \\
D & \text{interval on which } f(x) \text{ is increasing}
\end{align*}
\]

\[ \text{(A) } 1/2 \quad \text{(B) } 1 \quad \text{(C) } 2 \quad \text{(D) } 1/2, 2 \quad \text{(E) } -2 \quad \text{(F) } 0 \quad \text{(G) } -1 \quad \text{(H) } \text{Does not exist} \quad \text{(I) } 3/2 \quad \text{(J) } (1, 3) \]