Math 412, Analysis, Fall 2006

Problem Set 1, due Friday, September 29

1. 7: Which of the following sets is countable and which is not (provide detailed justification for your answers):

   (a) the set of irrational numbers

   We know that $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$, where $\mathbb{I}$ is the set of irrationals. Since $\mathbb{Q}$ is countable (as demonstrated in the text), and the union of countable sets is countable (theorem 1.2), it follows that if $\mathbb{I}$ is countable, then $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ is countable as well. This is a contradiction because $\mathbb{R}$ is uncountable (as shown in the text). We conclude that $\mathbb{I}$ is uncountable.

   (b) the set of terminating decimals

   A terminating decimal is a number which can be expressed as $\sum_{i=0}^{t} a_i 10^i + \sum_{j=1}^{k} b_j 10^{-j}$ for $a_1, \ldots, a_t, b_1, \ldots, b_k \in \{0, \ldots, 9\}$. Such a number is evidently rational, and there are certainly infinitely many of them. We conclude that the set of terminating rationals is an infinite subset of $\mathbb{Q}$, and as such, it is countable.

   (c) the set of real numbers between 0.357 and 0.358

   Let T be the set of real numbers between 0.357 and 0.358, let

$$S = \{0.3571 + \sum_{i=1}^{\infty} a_i/10^{i+5} \mid a_i \in \{0, 1\}\},$$

   let P be the set of all infinite sequences of zeros and ones, and let $f : P \to S$ be the function given by $f(\{s_i\}_{i=1}^{\infty}) = 0.3571 + \sum_{i=1}^{\infty} s_i/10^{i+5}$. Of course we have that $S \subset T$, and because decimal representation is unique, that f is a bijection. We showed in class that P is uncountable, and thus T contains an uncountable set (namely S). The contrapositive to the theorem which states that any infinite subset of a countable set is countable now implies that T is uncountable.

   (d) $\mathbb{Q} \times \mathbb{Q}$

   Since $\mathbb{Q}$ is countable, and the cross product of countable sets is countable, it follows that $\mathbb{Q} \times \mathbb{Q}$ is countable.

   (e) the set of numbers obtained from $\sqrt{2}$ and $\sqrt{3}$ by finitely many arithmetic operations $(+, -, \times, \div)$

   Let A be the set of numbers obtained from $\sqrt{2}$ and $\sqrt{3}$ by finitely many arithmetic operations $(+, -, \times, \div)$. We can describe each element $a \in A$ with a finite sequence made up of elements of the set $M = \{(,), \sqrt{2}, \sqrt{3}, +, -, \times, \div\}$. Now for each $a \in A$ choose a unique sequence s_a, and, writing S to be the set of all finite sequences on M, let $f : A \to S$ be the map $f(a) = s_a$. Clearly this is an injection, so that $\operatorname{im}(f)$ is an infinite subset of M. Since $\operatorname{card}(\operatorname{im}(f)) = \operatorname{card}(A)$, it is enough to show that S is countable. To demonstrate that S is countable, it is enough to show that the countable union of disjoint finite sets is countable.

   We do this in general. Let $B = \cup_{i \in I} B_i$ where the B_i are finite and disjoint and I is a countable set. Since $B \neq \emptyset$ is clear, we assume that B is infinite whence it is enough to show that B is countable. There is a natural injection from $\mathbb{N}$ to B. Let $f : \mathbb{N} \to I$ be a bijection, choose $b_i \in B_i$ for each $i \in I$, and let $g : \mathbb{N} \to B$ be the map $g(i) = b_{g(i)}$. By the Schroeder-Bernstein theorem, it is enough to demonstrate an injection from B to $\mathbb{N}$. Since each B_i is finite, we can choose a bijection $g_i : \{1, \ldots, t_{B_i}\} \to B_i$ (where $t_{B_i} \in \mathbb{N}$) for each $i \in I$. Note that $t_{B_{f(1)}} + \cdots + t_{B_{f(n)}}$ is the number of elements in

$$\bigcup_{\{f(i) \mid 1 \leq i \leq n\}} B_j,$$

   and we call this number T_n. Now let $g : B \to \mathbb{N}$ be given by $g(b) = T_n + (g_{f(j)})^{-1}(b)$ if $b \in B_{f(i)}$. Since the B_i are disjoint, g is well defined. To show that g is an injection, note that if $b, c \in B_{f(j)}$, then $g(b) \neq g(c)$ because $(g_{f(j)})^{-1}$ is an injection. If $b \in B_{f(i)}$ and $c \in B_{f(k)}$ for $j \neq k$, then without loss of generality assume that $j < k$. Now $T_j + g^{-1}_{f(j)}(b) \leq T_j + t_{B_{f(j)}} = T_{j+1} < T_k + 1 \leq T_k + g^{-1}_{f(k)}(c)$, so that $g(b) \neq g(c)$ as required. This completes the proof.

(f) $\mathbb{N} \times \mathbb{Z}$

Both $\mathbb{N}$ and $\mathbb{Z}$ are countable, ($\mathbb{N}$ is countable by definition, and $\mathbb{Z}$ is shown to be countable in the text). Since the cross product of countable sets is countable, we conclude that $\mathbb{N} \times \mathbb{Z}$ is countable.

(g) $\mathbb{R} \times \mathbb{Z}$

If $\mathbb{R} \times \mathbb{Z}$ is countable, than any subset is empty, finite, or countable. But consider the subset $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{Z}$. There is an obvious bijection from $\mathbb{R}$ to $\mathbb{R} \times \{0\}$, so that $\text{card}(\mathbb{R}) = \text{card}(\mathbb{R} \times \{0\})$, that is, $\mathbb{R}$ is either empty, finite, or countable. This gives a contradiction ($\mathbb{R}$ is neither finite nor empty, and we know from the text that it is not countable). We conclude that $\mathbb{R} \times \mathbb{Z}$ is not countable.

2. 16: Consider the set S of all real numbers obtained by taking rational powers of rational numbers. Is this set countable or uncountable?

   Let S be the set of real numbers obtained by taking rational powers of rational numbers and note that S is infinite (because $q^1 = q \in S$ for all $q \in \mathbb{Q}$). We will be finished if we can find a bijection from S to a countable set. Now for each $s \in S$, choose $q_s, p_s \in \mathbb{Q}$ such that $s = p_s^{q_s}$, and let $f : S \to \mathbb{Q} \times \mathbb{Q}$ be the map $f(s) = (p, q)$. Note that f is well defined because we have chosen unique representatives for each $s \in S$; f is injective because if $f(s) = f(s')$, then $(p_s, q_s) = (p_{s'}, q_{s'})$, whence $p_s = p_{s'}$ and $q_s = q_{s'}$, so that $s = p_s^{q_s} = p_{s'}^{q_{s'}} = s'$. It follows that f is a bijection from the infinite set S to an infinite subset of $\mathbb{Q} \times \mathbb{Q}$. Since $\mathbb{Q} \times \mathbb{Q}$ is countable, we know that any infinite subset is countable as well. This completes the proof.

3. 20: Let S be an infinite set. Prove that there is a subset $T \subseteq S$ such that T is countable.

   If S is countable, the $T = S$ is a countable (non-proper) subset of S. If S is uncountable, then it is infinite. Now, we can always choose an injection from $\mathbb{N}$ to any infinite set (simply inductively choose $s_i \in S - \{s_1, \dots, s_{i-1}\}$; here $S - \{s_1, \dots, s_{i-1}\}$ is never empty because S is not finite; the function $g : \mathbb{N} \to S$ by $g(i) = s_i$ is an injection). Thus $\text{card}(S) \geq \text{card}(\mathbb{N})$, and since we have assumed that S is uncountable, then it must be that $\text{card}(S) > \text{card}(\mathbb{N})$. Now let $f : \mathbb{N} \to S$ be an injection and let $T = \text{im}(f)$. It is immediate that f is a bijection from $\mathbb{N}$ to $\text{im} f = T$, so that T and $\mathbb{N}$ have the same cardinality, that it, T is a countable subset of S. This completes the proof.

4. 27: Let S be the set of finite sequences if 0s and 1s. Is this set countable or uncountable?

   Let S_i consist of the sequences of S which have exactly i terms. Then the cardinality of S_i is 2^i (in particular, S_i is finite for each i), and $S_i \cap S_j = \emptyset$ for $i \neq j$ (because S_i and S_j consist of sequences of different lengths). We note that $S = \cup_{i=0}^{\infty} S_i$, so it is enough to show that the disjoint union of countably many finite sets is countable. We already proved this above (in 7e), so the proof is complete.

5. 10: Let f be a function with domain the reals and range the reals. Assume that f has a local minimum at each point x in its domain. (This means that for each $x \in \mathbb{R}$ there is an $\epsilon > 0$ such that whenever $|x - t| < \epsilon$, then $f(x) \leq f(t)$). Prove that the image of f is countable.

   This should say, prove that the image of f is countable or finite (since the function $f = 1$ fulfills the requirements, but has a finite image).

   Let $S_n = \{y \in \mathbb{R} \mid \text{there is } x \in \mathbb{R} \text{ with } f(x) = y \text{ and } f(x) \leq f(t) \text{ for all } |x - t| < 1/n\}$. Note that $\text{im}(f) = \cup_{n \in \mathbb{N}} S_n$. This follows because for each $y \in \text{im}(f)$, there is some $x \in \mathbb{R}$ such that $f(x) = y$ and there is some $\epsilon > 0$ such that $f(x) \leq f(t)$ whenever $|x - t| < \epsilon$. Consider the real numbers $1/\epsilon$. By the Archimedian property, we may choose $P \in \mathbb{N}$ such that $P = P \cdot 1 > (1/\epsilon)$. Thus $1/P < \epsilon$, and hence $y \in S_P$. (I would have accepted it if you simply asserted that there exist $P \in \mathbb{N}$ such that $1/P < \epsilon$, and thus $y \in S_P$). One should note that the image of f is not empty.

   We next show that for each $n \in \mathbb{N}$, S_i is empty, finite, or countable. So assume that S_n is neither empty nor finite. For each $y \in S_n$ choose $x_y \in \mathbb{R}$ such that $f(x_y) = y$ and $f(x_y) \leq f(t)$ whenever $|x_y - t| < 1/n$. Now suppose that $|x_y - x_{y'}| < 1/n$ for some $y, y' \in S_n$ such that

$y \neq y'$. Since $f(x_y) \leq f(t)$ for all $|x_y - t| < 1/n$ and $f(x_{y'}) \leq f(t)$ for all $|x_{y'} - t| < 1/n$, it follows that $y = f(x_y) = f(x_{y'}) = y'$, a contradiction. Thus $|x_y - x_{y'}| \geq 1/n$ for all $y, y' \in S_n$ such that $y \neq y'$.

Now for each y, let q_y be a rational element in the interval $(x_y, x_y + 1/n)$ (such a rational exists because the rationals are dense in $\mathbb{R}$). Note that for $y, y' \in S$ such that $y \neq y'$, $q_y \neq q_{y'}$ (if $q_y = q_{y'}$, then $(x_y, x_y + 1/n) \cap (x_{y'}, x_{y'} + 1/n) \neq \emptyset$, and thus $|x_y - x_{y'}| < 1/n$, a contradiction). So let $g : S_n \to \mathbb{Q}$ be the map $g(y) = q_y$. This map is injective by construction, and hence gives a bijection from S_n to an infinite subset of $\mathbb{Q}$ (that the subset is infinite follows because S_n is infinite). We know that infinite subsets of countable sets are countable, and thus S_n is countable.

We will be finished if we can show that a countable collection of sets, each of which is either empty, finite, or countable, is either finite, empty, or countable.

Suppose that $B = \cup_{i \in I} B_i$ is a countable union and B_i is empty or finite. Note that B is in bijective correspondence with the set $\cup_{i \in I} (\cup_{b \in B_i} \{b, i\})$. Since $\cup_{i \in I} (\cup_{b \in B_i} \{b, i\})$ is a countable union of disjoint empty or finite sets, it is empty, finite, or countable (as we demonstrated in 7e). So we may assume that some B_i is countable. Let $I = I_1 \cup I_2 \cup I_3$ where B_i is empty for all $i \in I_1$, B_i is finite for all $i \in I_2$ and B_i is countable for all $i \in I_3$. Now $\cup_{i \in I_3} B_i$ is countable (as shown in the text), $\cup_{i \in I_1} B_i$ is empty, and $\cup_{i \in I_2} B_i$ is finite or countable (as shown in 7e). If $\cup_{i \in I_2} B_i$ is countable, then $\cup_{i \in I} B_i = (\cup_{i \in I_1} B_i) \cup (\cup_{i \in I_2} B_i) \cup (\cup_{i \in I_3} B_i) = (\cup_{i \in I_2} B_i) \cup (\cup_{i \in I_3} B_i)$ is countable, being the union of countable sets. So it remains to consider the case when $\cup_{i \in I_2} B_i$ is finite. That is, we need to demonstrate that if A is a finite set and B is a countable set, then $A \cup B$ is countable. This is, in fact, relatively obvious. Let $A' = A - B$. It is clear that A' is finite (if nonempty) and that $A \cup B = A' \cup B$. Now suppose that $f : \{1, \ldots, n\} \to A'$ and $g : \mathbb{N} \to B$ are bijections. The map $h : \mathbb{N} \to A' \times B$ by

$$h(i) = \begin{cases} f(i) & \text{for } 1 \leq i \leq n \\ g(i-n) & \text{for } i > n \end{cases}$$

is thus a bijection, and this completes the proof.

6. 14: Let S be the set of all subsets of the real numbers. Say that $X \in S$ is related to $Y \in S$ if $\text{card}(X) = \text{card}(Y)$. Is this an equivalence relation on S?

   Write $X \sim Y$ if X is related to Y. In order to demonstrate that $\sim$ is an equivalence relation, we must show that:

   (a) if $X \in S$, then $X \sim X$

   (b) if $Y, X \in S$ such that $X \sim Y$, then $Y \sim X$, and

   (c) if $X, Y, Z \in S$ such that $X \sim Y$ and $Y \sim Z$, then $X \sim Z$.

   By the definition of $\sim$, this is equivalent to showing that:

   (a) $\text{card}(X) = \text{card}(X)$ for all $x \in S$,

   (b) if $X, Y \in S$ and $\text{card}(X) = \text{card}(Y)$ then $\text{card}(X) = \text{card}(Y)$, and

   (c) if $X, Y, Z \in S$, $\text{card}(X) = \text{card}(Y)$, and $\text{card}(Y) = \text{card}(Z)$, then $\text{card}(X) = \text{card}(Z)$.

   Because $\text{card}(X) = \text{card}(Y)$ if and only if there is a bijection $f : X \to Y$, this is equivalent to showing that:

   (a) there is a bijection from X to X for all $X \in S$,

   (b) if $X, Y \in S$ and there is a bijection from X to Y, then there is a bijection from Y to X, and

   (c) if $X, Y, Z \in S$, there is a bijection from X to Y, and there is a bijection from Y to Z, then there is a bijection from X to Z.

   The identity map is a bijection from X to X, so that (6a) follows immediately. If $f : X \to Y$ is a bijection, then $f^{-1} : Y \to X$ is a bijection as well, hence (6b) is immediate. Now suppose that $f : X \to Y$ and $g : Y \to Z$ are bijections. It is easy to see that $g \circ f : X \to Z$ must also be a bijection, and hence (6c) is proved. We conclude that $\sim$ is an equivalence relation.

7. Give a careful proof by induction that $A_1 \times \cdots \times A_t$ is countable if A_i is countable for $i = 1, \ldots, t$.

   If $t = 1$, then the result is obvious, so suppose that $t > 1$ and that $A_1, \ldots, A_t$ are countable sets. Now $A_1 \times \cdots \times A_{t-1}$ is countable by the induction hypothesis, and thus $(A_1 \times \cdots A_{t-1}) \times A_t$ is countable (by proposition 1.2 in the text). This completes the proof since $(A_1 \times \cdots A_{t-1}) \times A_t = A_1 \times \cdots \times A_t$.