Special resolutions and the Eisenbud-Green-Harris conjecture

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Let $1 \leq a_1 \leq \cdots \leq a_n$ be integers in $\mathbb{N}$. Then we call L an $\{a_1, \ldots, a_n\}$ *lex-plus-powers ideal* if:

1. L is a monomial ideal minimally generated by $x_1^{a_1}, \ldots, x_n^{a_n}, m_1, \ldots, m_l$, and

2. for each $i = 1, \ldots, l$, if $r \in R_{\deg(m_i)}$ and $r \geq m_i$, then $r \in L$.

We say that such an ideal L is lex-plus-powers with respect to $\mathbb{A} = \{a_1, \ldots, a_n\}$.

x_1, x_2, x_3, x_4, x_5

$x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_3^2$
$x_3x_4, x_3x_5, x_4^2, x_4x_5, x_5^2$

$x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1^2x_5, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_2x_5,$
$x_1x_3^2, x_1x_3x_4, x_1x_3x_5, x_1x_4^2, x_1x_4x_5, x_1x_5^2, x_2^3, x_2^2x_3, x_2^2x_4,$
$x_2^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_4^2, x_2x_4x_5, x_2x_5^2, x_3^3, x_3^2x_4,$
$x_3^2x_5, x_3x_4^2, x_3x_4x_5, x_3x_5^2, x_4^3, x_4^2x_5, x_4x_5^2, x_5^3$

$x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^3x_5, x_1^2x_2^2, x_1^2x_2x_3, x_1^2x_2x_4, x_1^2x_2x_5,$
$x_1^2x_3^2, x_1^2x_3x_4, x_1^2x_3x_5, x_1^2x_4^2, x_1^2x_4x_5, x_1^2x_5^2, x_1x_2^3, x_1x_2^2x_3,$
$x_1x_2^2x_4, x_1x_2^2x_5, x_1x_2x_3^2, x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_4^2,$
$x_1x_2x_4x_5, x_1x_2x_5^2, x_1x_3^3, x_1x_3^2x_4, x_1x_3^2x_5, x_1x_3x_4^2,$
$x_1x_3x_4x_5, x_1x_3x_5^2, x_1x_4^3, x_1x_4^2x_5, x_1x_4x_5^2, x_1x_5^3, x_2^4, x_2^3x_3,$
$x_2^3x_4, x_2^3x_5, x_2^2x_3^2, x_2^2x_3x_4, x_2^2x_3x_5, x_2^2x_4^2, x_2^2x_4x_5, x_2^2x_5^2,$
$x_2x_3^3, x_2x_3^2x_4, x_2x_3^2x_5, x_2x_3x_4^2, x_2x_3x_4x_5, x_2x_3x_5^2, x_2x_4^3,$
$x_2x_4^2x_5, x_2x_4x_5^2, x_2x_5^3, x_3^4, x_3^3x_4, x_3^3x_5, x_3^2x_4^2, x_3^2x_4x_5, x_3^2x_5^2,$
$x_3x_4^3, x_3x_4^2x_5, x_3x_4x_5^2, x_3x_5^3, x_4^4, x_4^3x_5, x_4^2x_5^2, x_4x_5^3, x_5^4$

The Lex Plus Powers Conjectures:

Let $\mathcal{H}$ be a Hilbert function, $\mathbb{A}$ be a sequence of degrees, and suppose that there exists an ideal attaining $\mathcal{H}$ and minimally containing an $\mathbb{A}$-regular sequence (that is, it contains a regular sequence in the degrees $\mathbb{A}$, but in no smaller degrees).

1. There is a lex-plus-powers ideal with respect to $\mathbb{A}$ which attains $\mathcal{H}$. (Eisenbud, Green, Harris)

2. The lex-plus-powers ideal with respect to $\mathbb{A}$ has the largest graded Betti numbers among all ideals attaining $\mathcal{H}$ and containing a regular sequence in degrees $\mathbb{A}$.
(The lex plus powers conjecture—
Charalambous, Evans)

These conjectures follow the form of Macaulay's theorem and the Bigatti-Hulett-Pardue theorem (as well as implying them).

Given a Hilbert function $\mathcal{H}$:

- There is a lex ideal attaining $\mathcal{H}$.
  (Macaulay's theorem)

- The lex ideal attaining $\mathcal{H}$ has largest graded Betti numbers among all ideal attaining $\mathcal{H}$.
  (Bigatti-Hulett-Pardue theorem)

The Eisenbud-Green-Harris conjecture is known for:

- ideals containing the powers of the variables [Clements, Lindstrom],
- dimension 2,
- Certain cases in dimension 3 [Cooper],
- dimension ≤ 5 if there is a maximal regular sequence in degree 2.

Equivalent formulations:

- [EGH] Suppose that I is an ideal containing a regular sequence in degrees $\mathbb{A}$, and L is a $\mathbb{A}$ lex-plus-powers ideal such that

$$H(R/I,d) = H(R/L,d).$$

  Then

$$H(R/I,d+1) \leq H(R/L_{\leq d},d+1).$$

- [Generator version] Suppose that I is an ideal containing a regular sequence in degrees $\mathbb{A}$, and L is an $\mathbb{A}$ lex-plus-powers ideal such that $H(R/I) = H(R/L)$. Then $\beta^I_{1,j} \leq \beta^L_{1,j}$ for all j.

- [Socle version] Suppose that I is an ideal containing a regular sequence in degrees $\mathbb{A}$, and L is an $\mathbb{A}$ lex-plus-powers ideal such that $H(R/I) = H(R/L)$. Then $\beta^I_{n,j} \leq \beta^L_{n,j}$ for all j.

- [Socle version 2] Suppose that I is an ideal containing a regular sequence in degrees $\mathbb{A}$, and L is an $\mathbb{A}$ lex-plus-powers ideal such that $H(R/I) = H(R/L)$. Then $\beta^I_{n,\rho+n-1} \leq \beta^L_{n,\rho+n-1}$ (here ρ is the regularity of $H(R/L)$).

β^L	α_0^L	α_1^L	$\dots$	α_n^L
0	1	$\square$	$\dots$	$\square$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\rho - 2$	0	$\square$	$\dots$	$\square$
$\rho - 1$	0	$\square$	$\dots$	$\square$
ρ	0	$\square$	$\dots$	$=$

β^I	α_0^I	α_1^I	$\dots$	α_n^I
0	1	$\square$	$\dots$	$\square$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\rho - 2$	0	$\square$	$\dots$	$\square$
$\rho - 1$	0	$\square$	$\dots$	$\square$
ρ	0	$\square$	$\dots$	$=$

Special classes of counter examples:

If EGH fails, then there is an ideal I with the following properties:

- I contains an $\mathbb{A}$-regular sequence, $\{f_1, \ldots, f_n\}$,
- $H(R/I) = H(R/L)$ where L is an $\mathbb{A}$ lex-plus-powers ideal,
- $\beta^I_{n,\rho+n-1} > \beta^L_{n,\rho+n-1}$ (where ρ is the regularity of $H(R/I)$),
- $I_{\leq \rho-1} = (f_1, \ldots, f_n)_{\leq \rho-1}$,
- $L_{\leq \rho-1} = (x_1^{a_1}, \ldots, x_n^{a_n})_{\leq \rho-1}$.

β^L	α_0^L	α_1^L	α_2^L	$\ldots$	α_{n-1}^L	α_n^L
0	1	$\square$	$\square$	$\ldots$	$\square$	$\square$
1	0	$\square$	$\square$	$\ldots$	$\square$	$\square$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\rho - 2$	0	$\square$	$\square$	$\ldots$	$\square$	$\square$
$\rho - 1$	0	$=$	$?$	$\ldots$	$?$	$<$
ρ	0	$?$	$?$	$\ldots$	$?$	$=$

β^I	α_0^I	α_1^I	α_2^I	$\ldots$	α_{n-1}^I	α_n^I
0	1	$\square$	$\square$	$\ldots$	$\square$	$\square$
1	0	$\square$	$\square$	$\ldots$	$\square$	$\square$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\rho - 2$	0	$\square$	$\square$	$\ldots$	$\square$	$\square$
$\rho - 1$	0	$=$	$?$	$\ldots$	$?$	$>$
ρ	0	$?$	$?$	$\ldots$	$?$	$=$

Any ideal with such a Betti diagram can be generated by a generic sequence of forms (but not necessarily minimally generated by such a sequence).

We say that $g_1, \ldots, g_r$ is a generic sequence of forms if the multiplication

$$g_i : \Big(R/(g_1, \ldots, g_{i-1})\Big)_d \to \Big(R/(g_1, \ldots, g_{i-1})\Big)_{d+\deg g_i}$$

is either injective or surjective for all degrees d and all $i = 1, \ldots, r$.

In particular, if we consider the set of sequences $g_1, \ldots, g_r$ as an affine space where the coordinates give the coefficients of the polynomials, then the set of generic sequences defines a Zariski open set.

There are some theorems about ideals generated by generic sequences which might be useful here.

If EGH fails, then there is an ideal I with the following properties:

- I contains an $\mathbb{A}$-regular sequence,
- $H(R/I) = H(R/L)$ where L is an $\mathbb{A}$ lex-plus-powers ideal,
- $\beta^I_{1,\rho+1} > \beta^L_{1,\rho+1}$ (where ρ is the regularity of $H(R/I)$),
- $\beta^I_{1,j} \leq \beta^L_{1,j}$ for all $j \leq \rho$,
- R/I is level.

β^L	α_0^L	α_1^L	α_2^L	$\dots$	α_{n-1}^L	α_n^L
0	1	$=$	$=$	$\dots$	$=$	0
1	0	$\geq$	?	$\dots$	?	$\geq$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\rho - 2$	0	$\geq$	?	$\dots$	?	$\geq$
$\rho - 1$	0	$\geq$	?	$\dots$	?	$\geq$
ρ	0	$<$	?	$\dots$	?	$=$

β^I	α_0^I	α_1^I	α_2^I	$\dots$	α_{n-1}^I	α_n^I
0	1	$=$	$=$	$\dots$	$=$	0
1	0	$\leq$	?	$\dots$	?	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\rho - 2$	0	$\leq$	?	$\dots$	?	0
$\rho - 1$	0	$\leq$	?	$\dots$	?	0
ρ	0	$>$	?	$\dots$	?	$=$

This counterexample seems to be related to recent work of Geramita, Harima, Migliore, and Shin about the Hilbert functions of level algebras.

After all, EGH is a conjecture about possible Hilbert functions, and is still unknown in dimension 3.

At least one technique from Geramita, Harima, Migliore, and Shin's work looks promising.

Using Macaulay's inverse systems to reformulate EGH:

Let $S = k[y_1, \ldots, y_n]$ be considered as an R-module where the action of x_i on S is partial differentiation with respect to y_i.

There is a bijection between Artinian ideals $I \subseteq R$ and finitely generated R-submodules I^{-1} of S where $I^{-1} = \{s \in S \mid m \circ s = 0 \text{ for all m} \in \text{I}\}$.

For all d, $H(I^{-1}, d) = H(R/I, d)$.

For all d, $\dim(\text{Soc}_{\text{R/I}}(\text{d}))$ is equal to the number of minimal generators of I^{-1} in degree d.

Suppose that L is $\mathbb{A}$ lex-plus-powers, write $\mathcal{L}$ to denote L^{-1} and let $M_{\mathbb{A}}(d)$ denote a monomial basis for the set $\{m \in S_d \mid y_i^{a_i}$ divides m for some i$\}$. Then

1. $\mathcal{L}_d \cap M_{\mathbb{A}}(d) = \emptyset$,

2. if $m \in \mathcal{L}_d$ and $m' \in S_d - M_{\mathbb{A}}(d)$ such that $m' < m$, then $m' \in \mathcal{L}$.

If $N \subseteq S_d$ satisfies 1 and 2 above, we say that N is SLpp(?) with respect to $\mathbb{A}$.

Another fact: suppose that I contains an $\mathbb{A}$-regular sequence $\{f_1, \ldots, f_n\}$, and write $M_{\underline{f}}(d)$ to denote a basis for the set

$$\{g \in S_d \mid \text{replacing the } y_i \text{ in } g \text{ with } x_i \text{ gives a polynomial in } (f_1, \ldots, f_n)_d\}.$$

Then $\mathcal{I}_d \cap M_{\underline{f}}(d) = \emptyset$.

Suppose that L is $\mathbb{A}$ lex-plus-powers, write $\mathcal{L}$ to denote L^{-1} and let $M_{\mathbb{A}}(d)$ denote the set $\{m \in S_d \mid y_i^{a_i}$ divides m for some i$\}$. Then

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$$\{g \in S_d \ \mid \ \text{replacing the } y_i \text{ in } g \text{ with } x_i \text{ gives a polynomial in } (f_1, \ldots, f_n)_d\}.$$

Then $\mathcal{I}_d \cap M_{\underline{f}}(d) = \emptyset$.

$$x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^3x_5, x_1^2x_2^2, x_1^2x_2x_3, x_1^2x_2x_4, x_1^2x_2x_5, x_1^2x_3^2, x_1^2x_3x_4, x_1^2x_3x_5, x_1^2x_4^2, x_1^2x_4x_5, x_1^2x_5^2, x_1x_2^3, x_1x_2^2x_3, x_1x_2^2x_4, x_1x_2^2x_5, x_1x_2x_3^2, x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_4^2, x_1x_2x_4x_5, x_1x_2x_5^2, x_1x_3^3, x_1x_3^2x_4, x_1x_3^2x_5, x_1x_3x_4^2, x_1x_3x_4x_5, x_1x_3x_5^2, x_1x_4^3, x_1x_4^2x_5, x_1x_4x_5^2, x_1x_5^3, x_2^4, x_2^3x_3, x_2^3x_4, x_2^3x_5, x_2^2x_3^2, x_2^2x_3x_4, x_2^2x_3x_5, x_2^2x_4^2, x_2^2x_4x_5, x_2^2x_5^2, x_2x_3^3, x_2x_3^2x_4, x_2x_3^2x_5, x_2x_3x_4^2, x_2x_3x_4x_5, x_2x_3x_5^2, x_2x_4^3, x_2x_4^2x_5, x_2x_4x_5^2, x_2x_5^3, x_3^4, x_3^3x_4, x_3^3x_5, x_3^2x_4^2, x_3^2x_4x_5, x_3^2x_5^2, x_3x_4^3, x_3x_4^2x_5, x_3x_4x_5^2, x_3x_5^3, x_4^4, x_4^3x_5, x_4^2x_5^2, x_4x_5^3, x_5^4$$

$$x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1^2x_5, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3^2, x_1x_3x_4, x_1x_3x_5, x_1x_4^2, x_1x_4x_5, x_1x_5^2, x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_4^2, x_2x_4x_5, x_2x_5^2, x_3^3, x_3^2x_4, x_3^2x_5, x_3x_4^2, x_3x_4x_5, x_3x_5^2, x_4^3, x_4^2x_5, x_4x_5^2, x_5^3$$

$$x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_3^2 \; x_3x_4, x_3x_5, x_4^2, x_4x_5, x_5^2$$

$$x_1, x_2, x_3, x_4, x_5$$

EGH is equivalent to the following:

Suppose

- $\{f_1, \ldots, f_n\}$ is an $\mathbb{A}$-regular sequence,
- $\mathcal{I} \subseteq S$ is such that $\mathcal{I} \cap M_{\underline{f}}(d) = \emptyset$ and the largest degree in which $\mathcal{I}$ has generators is d,
- $\mathcal{L} \subseteq S$ is SLpp with respect to $\mathbb{A}$ and the largest degree in which $\mathcal{L}$ has generators is d,
- $H(\mathcal{L}, d) = H(\mathcal{I}, d)$,

Then $H(\mathcal{L}_{\leq d}, d-1) \leq H(\mathcal{I}, d-1)$.

In this formulation the problem is reduced to understanding the growth of a module in the first nonzero degree.

Hope: this approach will provide a non-combinatorial proof of EGH for monomial ideals in dimension three.