The Eisenbud-Green-Harris Conjecture

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October 3, 2004

Let $1 \leq a_1 \leq \cdots \leq a_n$ be integers in $\mathbb{N}$. Then we call L an $\{a_1, \ldots, a_n\}$ *lex-plus-powers ideal* if:

1. L is a monomial ideal minimally generated by $x_1^{a_1}, \ldots, x_n^{a_n}, m_1, \ldots, m_l$, and

2. for each $i = 1, \ldots, l$, if $r \in R_{\deg(m_i)}$ and $r \geq m_i$, then $r \in L$.

We say that such an ideal L is lex-plus-powers with respect to $\mathbb{A} = \{a_1, \ldots, a_n\}$.

$$x_1, x_2, x_3, x_4, x_5$$

$$x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_3^2$$
$$x_3x_4, x_3x_5, x_4^2, x_4x_5, x_5^2$$

$$x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1^2x_5, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_2x_5,$$
$$x_1x_3^2, x_1x_3x_4, x_1x_3x_5, x_1x_4^2, x_1x_4x_5, x_1x_5^2, x_2^3, x_2^2x_3, x_2^2x_4,$$
$$x_2^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_4^2, x_2x_4x_5, x_2x_5^2, x_3^3, x_3^2x_4,$$
$$x_3^2x_5, x_3x_4^2, x_3x_4x_5, x_3x_5^2, x_4^3, x_4^2x_5, x_4x_5^2, x_5^3$$

$$x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^3x_5, x_1^2x_2^2, x_1^2x_2x_3, x_1^2x_2x_4, x_1^2x_2x_5,$$
$$x_1^2x_3^2, x_1^2x_3x_4, x_1^2x_3x_5, x_1^2x_4^2, x_1^2x_4x_5, x_1^2x_5^2, x_1x_2^3, x_1x_2^2x_3,$$
$$x_1x_2^2x_4, x_1x_2^2x_5, x_1x_2x_3^2, x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_4^2,$$
$$x_1x_2x_4x_5, x_1x_2x_5^2, x_1x_3^3, x_1x_3^2x_4, x_1x_3^2x_5, x_1x_3x_4^2,$$
$$x_1x_3x_4x_5, x_1x_3x_5^2, x_1x_4^3, x_1x_4^2x_5, x_1x_4x_5^2, x_1x_5^3, x_2^4, x_2^3x_3,$$
$$x_2^3x_4, x_2^3x_5, x_2^2x_3^2, x_2^2x_3x_4, x_2^2x_3x_5, x_2^2x_4^2, x_2^2x_4x_5, x_2^2x_5^2,$$
$$x_2x_3^3, x_2x_3^2x_4, x_2x_3^2x_5, x_2x_3x_4^2, x_2x_3x_4x_5, x_2x_3x_5^2, x_2x_4^3,$$
$$x_2x_4^2x_5, x_2x_4x_5^2, x_2x_5^3, x_3^4, x_3^3x_4, x_3^3x_5, x_3^2x_4^2, x_3^2x_4x_5, x_3^2x_5^2,$$
$$x_3x_4^3, x_3x_4^2x_5, x_3x_4x_5^2, x_3x_5^3, x_4^4, x_4^3x_5, x_4^2x_5^2, x_4x_5^3, x_5^4$$

Defining lex-plus-powers ideals allows us to make the following conjectures.

Let $\mathcal{H}$ be a Hilbert function, $\mathbb{A}$ be a sequence of degrees, and suppose that there exists an ideal attaining $\mathcal{H}$ and minimally containing an $\mathbb{A}$-regular sequence (that is, a regular sequence in the degrees $\mathbb{A}$).

1. There is a lex-plus-powers ideal with respect to $\mathbb{A}$ which attains $\mathcal{H}$. (Eisenbud, Green, Harris)

2. The lex-plus-powers ideal with respect to $\mathbb{A}$ has the largest graded Betti numbers among all ideals attaining $\mathcal{H}$ and containing a regular sequence in degrees $\mathbb{A}$.
   (The lex-plus-powers conjecture—
   Charalambous, Evans)

These conjectures mimic Macaulay's theorem and the Bigatti-Hulett-Pardue theorem (as well as implying them).

Given a Hilbert function $\mathcal{H}$:

- There is a lex ideal attaining $\mathcal{H}$.
  (Macaulay's theorem)

- The lex ideal attaining $\mathcal{H}$ has largest graded Betti numbers among all ideal attaining $\mathcal{H}$.
  (Bigatti-Hulett-Pardue theorem)

(One needs to prove that the lex ideal attaining $\mathcal{H}$ contains the "latest" possible regular sequence, that is, if the lex ideal contains a regular sequence in degrees $\{a_1, \ldots, a_n\}$, then so does every ideal attaining $\mathcal{H}$).

A slightly broader context:

- If $\Omega \subset \mathbb{P}^n$ is a complete intersection of quadrics, then any hypersurface of degree k that contains a subscheme $\Gamma \subset \Omega$ of degree strictly greater than $2^n - 2^{n-k}$ must contain Ω. [Generalized Cayley-Bacharach conjecture]

- Suppose that I is an ideal containing a maximal regular sequence in degree 2 and
$$H(R/I,d) = \binom{a_d}{d} + \cdots + \binom{a_1}{1}$$
is the d-th Macaulay representation of $H(R/I,d)$. Then
$$H(R/I,d+1) \leq \binom{a_d}{d+1} + \cdots + \binom{a_1}{1+1}.$$

- Suppose that I is an ideal containing a maximal regular sequence in degree 2 and L is a $\{2,\ldots,2\}$ lex-plus- powers ideal such that
$$H(R/I,d) = H(R/L,d).$$
Then
$$H(R/I,d+1) \leq H(R/L_d,d+1).$$

- [EGH] Suppose that I is an ideal containing a regular sequence in degrees $\mathbb{A}$, and L is a $\mathbb{A}$ lex-plus-powers ideal such that

$$H(R/I,d) = H(R/L,d).$$

Then

$$H(R/I,d+1) \leq H(R/L_{\leq d},d+1).$$

- [Generator version] Suppose that I is an ideal containing a regular sequence in degrees $\mathbb{A}$, and L is an $\mathbb{A}$ lex-plus-powers ideal such that $H(R/I) = H(R/L)$. Then $\beta^I_{1,j} \leq \beta^L_{1,j}$ for all j.

- [Socle version] Suppose that I is an ideal containing a regular sequence in degrees $\mathbb{A}$, and L is an $\mathbb{A}$ lex-plus-powers ideal such that $H(R/I) = H(R/L)$. Then $\beta^I_{n,j} \leq \beta^L_{n,j}$ for all j.

- [Socle version 2] Suppose that I is an ideal containing a regular sequence in degrees $\mathbb{A}$, and L is an $\mathbb{A}$ lex-plus-powers ideal such that $H(R/I) = H(R/L)$. Then $\beta^I_{n,\rho+n-1} \leq \beta^L_{n,\rho+n-1}$ (here ρ is the regularity of $H(R/L)$).

β^L	α_0^L	α_1^L	$\dots$	α_n^L
0	1	$\square$	$\dots$	$\square$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\rho - 2$	0	$\square$	$\dots$	$\square$
$\rho - 1$	0	$\square$	$\dots$	$\square$
ρ	0	$\square$	$\dots$	$=$

β^I	α_0^I	α_1^I	$\dots$	α_n^I
0	1	$\square$	$\dots$	$\square$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\rho - 2$	0	$\square$	$\dots$	$\square$
$\rho - 1$	0	$\square$	$\dots$	$\square$
ρ	0	$\square$	$\dots$	$=$

The Generalized Cayley-Bacharach conjecture is known for:

- The case for which $n \leq 7$. (Eisenbud, Green, Harris)

- The generalized hypercube, that is, the case for which Ω consists of the 2^n common zeros defined by n quadratics, each of which is a product of linear forms. (Evans, Riehl)

The Eisenbud-Green-Harris conjecture is known for:

- ideals containing the powers of the variables [Clements, Lindstrom],
- dimension 2,
- dimension ≤ 5 if there is a maximal regular sequence in degree 2.

It is also known that minimal resolutions for lex-plus-powers ideals can be computed by:

- removing elements divisible by $x_i^{a_i}$ for $i > 1$ from the Eliahou-Kervaire resolution on L,
- then iteratively using colon ideals and the mapping cone to introduce terms corresponding to the pure powers. (Charalambous, Evans)

Special classes of counter examples:

If EGH fails, then there is an ideal I with the following properties:

- I contains an $\mathbb{A}$-regular sequence, $\{f_1, \ldots, f_n\}$,

- $H(R/I) = H(R/L)$ where L is an $\mathbb{A}$ lex-plus-powers ideal,

- $\beta^I_{n,\rho+n-1} > \beta^L_{n,\rho+n-1}$ (where ρ is the regularity of $H(R/I)$),

- $I_{\leq \rho-1} = (f_1, \ldots, f_n)_{\leq \rho-1}$,

- $L_{\leq \rho-1} = (x_1^{a_1}, \ldots, x_n^{a_n})_{\leq \rho-1}$.

β^L	α_0^L	α_1^L	α_2^L	$\ldots$	α_{n-1}^L	α_n^L
0	1	$\square$	$\square$	$\ldots$	$\square$	$\square$
1	0	$\square$	$\square$	$\ldots$	$\square$	$\square$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\rho - 2$	0	$\square$	$\square$	$\ldots$	$\square$	$\square$
$\rho - 1$	0	$=$	$?$	$\ldots$	$?$	$<$
ρ	0	$?$	$?$	$\ldots$	$?$	$=$

β^I	α_0^I	α_1^I	α_2^I	$\ldots$	α_{n-1}^I	α_n^I
0	1	$\square$	$\square$	$\ldots$	$\square$	$\square$
1	0	$\square$	$\square$	$\ldots$	$\square$	$\square$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\rho - 2$	0	$\square$	$\square$	$\ldots$	$\square$	$\square$
$\rho - 1$	0	$=$	$?$	$\ldots$	$?$	$>$
ρ	0	$?$	$?$	$\ldots$	$?$	$=$

If EGH fails, then there is an ideal I with the following properties:

- I contains an $\mathbb{A}$-regular sequence,
- $H(R/I) = H(R/L)$ where L is an $\mathbb{A}$ lex-plus-powers ideal,
- $\beta^I_{1,\rho+1} > \beta^L_{1,\rho+1}$ (where ρ is the regularity of $H(R/I)$),
- $\beta^I_{1,j} \leq \beta^L_{1,j}$ for all $j \leq \rho$,
- R/I is level.

β^L	α_0^L	α_1^L	α_2^L	$\dots$	α_{n-1}^L	α_n^L
0	1	$=$	$=$	$\dots$	$=$	0
1	0	$\geq$	?	$\dots$	?	$\geq$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\rho - 2$	0	$\geq$	?	$\dots$	?	$\geq$
$\rho - 1$	0	$\geq$	?	$\dots$	?	$\geq$
ρ	0	$<$	?	$\dots$	?	$=$

β^I	α_0^I	α_1^I	α_2^I	$\dots$	α_{n-1}^I	α_n^I
0	1	$=$	$=$	$\dots$	$=$	0
1	0	$\leq$	?	$\dots$	?	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\rho - 2$	0	$\leq$	?	$\dots$	?	0
$\rho - 1$	0	$\leq$	?	$\dots$	?	0
ρ	0	$>$	?	$\dots$	?	$=$

Using Macaulay's inverse systems to control socles:

Let $S = k[y_1, \ldots, y_n]$ be considered as an R-module where the action of x_i on S is partial differentiation with respect to y_i.

There is a bijection between Artinian ideals $I \subseteq R$ and finitely generated R-submodules I^{-1} of S where $I^{-1} = \{s \in S \mid m \circ s = 0 \text{ for all m} \in \mathrm{I}\}$.

For all d, $H(I^{-1}, d) = H(R/I, d)$.

For all d, $\dim(\mathrm{Soc}_{\mathrm{R/I}}(\mathrm{d}))$ is equal to the number of minimal generators of I^{-1} in degree d.

Transfer the conjecture to S.

Suppose that L is $\mathbb{A}$ lex-plus-powers, write $\mathcal{L}$ to denote L^{-1} and let $M_{\mathbb{A}}(d)$ denote a monomial basis for the set $\{m \in S_d \mid y_i^{a_i}$ divides m for some i$\}$. Then

1. $\mathcal{L}_d \cap M_{\mathbb{A}}(d) = \emptyset$,

2. if $m \in \mathcal{L}_d$ and $m' \in S_d - M_{\mathbb{A}}(d)$ such that $m' < m$, then $m' \in \mathcal{L}$.

If $N \subseteq S_d$ satisfies 1 and 2 above, we say that N is SLpp(?) with respect to $\mathbb{A}$.

$$x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^3x_5, x_1^2x_2^2, x_1^2x_2x_3, x_1^2x_2x_4, x_1^2x_2x_5, x_1^2x_3^2, x_1^2x_3x_4, x_1^2x_3x_5, x_1^2x_4^2, x_1^2x_4x_5, x_1^2x_5^2, x_1x_2^3, x_1x_2^2x_3, x_1x_2^2x_4, x_1x_2^2x_5, x_1x_2x_3^2, x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_4^2, x_1x_2x_4x_5, x_1x_2x_5^2, x_1x_3^3, x_1x_3^2x_4, x_1x_3^2x_5, x_1x_3x_4^2, x_1x_3x_4x_5, x_1x_3x_5^2, x_1x_4^3, x_1x_4^2x_5, x_1x_4x_5^2, x_1x_5^3, x_2^4, x_2^3x_3, x_2^3x_4, x_2^3x_5, x_2^2x_3^2, x_2^2x_3x_4, x_2^2x_3x_5, x_2^2x_4^2, x_2^2x_4x_5, x_2^2x_5^2, x_2x_3^3, x_2x_3^2x_4, x_2x_3^2x_5, x_2x_3x_4^2, x_2x_3x_4x_5, x_2x_3x_5^2, x_2x_4^3, x_2x_4^2x_5, x_2x_4x_5^2, x_2x_5^3, x_3^4, x_3^3x_4, x_3^3x_5, x_3^2x_4^2, x_3^2x_4x_5, x_3^2x_5^2, x_3x_4^3, x_3x_4^2x_5, x_3x_4x_5^2, x_3x_5^3, x_4^4, x_4^3x_5, x_4^2x_5^2, x_4x_5^3, x_5^4$$

$$x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1^2x_5, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3^2, x_1x_3x_4, x_1x_3x_5, x_1x_4^2, x_1x_4x_5, x_1x_5^2, x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_4^2, x_2x_4x_5, x_2x_5^2, x_3^3, x_3^2x_4, x_3^2x_5, x_3x_4^2, x_3x_4x_5, x_3x_5^2, x_4^3, x_4^2x_5, x_4x_5^2, x_5^3$$

$$x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2^2, x_2x_3, x_2x_4, x_2x_5, x_3^2 \; x_3x_4, x_3x_5, x_4^2, x_4x_5, x_5^2$$

$$x_1, x_2, x_3, x_4, x_5$$

Another fact: suppose that I contains an $\mathbb{A}$-regular sequence $\{f_1, \ldots, f_n\}$, and write $M_{\underline{f}}(d)$ to denote a basis for the set

$$\{\sum_{j-1}^{q} y_1^{\alpha_{1,j}} \cdots y_n^{\alpha_{n,j}} \in S_d \mid \sum_{j=1}^{q} x_1^{\alpha_{1,j}} \cdots x_n^{\alpha_{n,j}} \in (f_1, \ldots, f_n)_d\}.$$

Then $\mathcal{I}_d \cap M_{\underline{f}}(d) = \emptyset$.

EGH is equivalent to the following:

Suppose

- $\{f_1, \ldots, f_n\}$ is an $\mathbb{A}$-regular sequence,
- $\mathcal{I} \subseteq S$ is such that $\mathcal{I} \cap M_{\underline{f}}(d) = \emptyset$ and the largest degree in which $\mathcal{I}$ has generators is d,
- $\mathcal{L} \subseteq S$ is SLpp with respect to $\mathbb{A}$ and the largest degree in which $\mathcal{L}$ has generators is d,
- $H(\mathcal{L}, d) = H(\mathcal{I}, d)$,

Then $H(\mathcal{L}_{\leq d}, d-1) \leq H(\mathcal{I}, d-1)$.

Hope: this approach will provide a non-combinatorial proof of EGH for monomial ideals (at least in dimension 3).

It does give a different perspective on how the degrees of the regular sequence affect the problem.