SUMMARY OF RESEARCH INTERESTS
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Many problems in commutative algebra concern free resolutions, and the invariants which arise from them. In the last three years I have had the opportunity to study free resolutions as a post doc at the University of Michigan, and have undertaken several individual and joint projects as well as supervising two undergraduate research projects. The result has been satisfying progress on some important and longstanding conjectures. Most of this work is ongoing, and I look forward to continuing these inquiries—joint, individual, and undergraduate—in a tenure track position.

Much of my recent interest in free resolutions has concerned their extremal properties. Let R be the polynomial ring in n variables over a field k and $I \subset R$ be $(x_1, \ldots, x_n)$-primary[1]. Then we can form a free resolution of R/I,

$$\mathcal{F} = 0 \to R^{\alpha_n^I} \to R^{\alpha_{n-1}^I} \to \cdots \to R^{\alpha_0^I} \to M \to 0$$

by, at the ith step, mapping α_i^I copies of R onto the kernel of the map created at the $(i-1)$st step[2]. If we insist that none of the entries in the matrices giving the resulting maps is a non-zero field element, then the α_i^I form an invariant called the Betti numbers of I. When I is homogeneous we can obtain a finer invariant by keeping track of degrees at each step. In this case we write a minimal free resolution of R/I as

$$0 \to \sum_j R[-j]^{\beta_{n,j}^I} \to \sum_j R[-j]^{\beta_{n-1,j}^I} \to \cdots \to \sum_j R[-j]^{\beta_{0,j}^I} \to R/I \to 0,$$

and the $\beta_{i,j}$ give the graded Betti numbers of R/I. Here we can also define the Hilbert function of R/I, which is the function $H(R/I, d) : \mathbb{N} \to \mathbb{N}$ such that $H(R/I, d) = \dim_k(R_d/I_d)$. So $H(R/I, d)$ gives the k-vector space dimension of the degree d forms in R which are not in I.

One way to study the extremal properties of free resolutions is to investigate the conditions under which the Betti numbers are bounded (above or below), and how this relates to the growth of the Hilbert function. In the following, I describe some conjectures about this type of extremal behavior, my contributions to these problems, and my future goals.

The Eisenbud, Green, Harris conjectures: It is known that no global upper bound for the Betti numbers exists. If we restrict to the graded case and fix a Hilbert function, however, then the graded Betti numbers of the lex ideal can be shown to give an upper bound

[1] I is $(x_1, \ldots, x_n)$-primary if $(x_1, \ldots, x_n)^t \subset I$ for some $t \gg 0$.

[2] In the general case, we do this for a regular local ring, and a finite length R-module M.

[B, Hu, P1]. In this case a sharp lower bounds need not exist[3] [CE1]. In [R1], I found an infinite family for which this is true, and demonstrated that a sharp lower bound may fail to exist even if the fixed Hilbert function is that of an R-sequence.

Eisenbud, Green, and Harris have conjectured that a finer bound holds if we restrict to ideals containing regular sequences in certain degrees. We need the following definition of Charalambous and Evans [CE2]: for $x_1 > \cdots > x_n$ and $>_{\text{lex}}$ the lex order on the monomials of R, a monomial ideal I is called an $\{a_1, \ldots, a_n\}$*-lex plus powers ideal* if I is minimally generated by $x_1^{a_1}, \ldots, x_n^{a_n}, m_1, \ldots, m_l$ where for each j, if $r \in R_{\deg(m_j)}$ and $r >_{\text{lex}} m_j$, then $r \in I$. We call a regular sequence $\underline{\boldsymbol{x}}$ an $\{a_1, \ldots, a_n\}$-regular sequence if the degrees of the terms of $\underline{\boldsymbol{x}}$ are $a_1, \ldots, a_n$. Then the Lex Plus Powers (LPP) conjecture of Eisenbud, Green, and Harris states that, given a Hilbert function $\mathcal{H}$, the $\{a_1, \ldots, a_n\}$-lex plus powers ideal which attains $\mathcal{H}$ has everywhere the unique largest graded Betti numbers among all ideals attaining $\mathcal{H}$ and containing an $\{a_1, \ldots, a_n\}$-regular sequence. This conjecture is intimately related to a weaker statement of Eisenbud, Green, and Harris, which claims that when the comparison can be made, $\{a_1, \ldots, a_n\}$-lex plus powers ideals give the largest possible Hilbert function growth among all ideals containing an $\{a_1, \ldots, a_n\}$-regular sequence [EGH].

Though these statements seem rather technical, one should not lose track of the analogy being made with the general result. In the general case, a certain lex ideal attains the unique largest resolution among all cyclic modules with a given Hilbert function. The Lex Plus Powers conjecture states that this structure on resolutions has a finer interpretation: among all cyclic modules with a given Hilbert function and containing a regular sequence of given degrees, a certain lex plus powers ideal attains the unique largest resolution. A lex ideal is, after all, just a special lex plus powers ideal. A similar statement can be made about the weaker form of the conjecture (in relation to Macaulay's theorem [M]).

The LPP conjecture also has broader implications. A special case of this problem implies a conjecture important to algebraic geometers, the generalized Cayley-Bacharach conjecture. The Cayley-Bacharach conjecture constrains the number of zeros a degree t polynomial can share with the zero set of n quadratics without vanishing on the whole set[4].

It turns out that very little is known about these conjectures. My own contributions occur in two papers. In [ER] Evans and I gave evidence that LPP holds by exploring all the resolutions which could occur for a specific Hilbert function (in that paper I also showed that after fixing a Hilbert function, the degrees of the generators of the ith syzygy module place certain restraints on the smallest degree in which the $(i+1)$st syzygy can have a generator). In [R2] I proved that if the weaker conjecture holds, then lex plus powers ideals also have largest socles. I also gave several equivalent forms of the weaker conjecture, in particular

[3] The global lower bound $\beta_{i,j} = 0$ for all i, j always exists

[4] The conjecture claims that if $\Omega \subset \mathbb{P}^r$ is a complete intersection of quadrics, then any hypersurface of degree t that contains a subscheme $\Gamma \subset \Omega$ of degree strictly greater than $2^r - 2^{r-t}$ must contain Ω.

that the weaker conjecture implies that after fixing a Hilbert function $\mathcal{H}$ and a set of degrees $\{a_1, \ldots, a_n\}$, then the $\{a_1, \ldots, a_n\}$-lex plus powers ideal attaining $\mathcal{H}$ has the most generators in each degree (when compared with other ideals attaining $\mathcal{H}$ and containing an $\{a_1, \ldots, a_n\}$-regular sequence). Thus, using the terminology of Macaulay 2, the result on socles means that if the first columns of the Betti diagrams of lex plus powers ideals are known to be largest, then the final columns are largest as well. This is a first step in showing that the weaker conjecture and LPP are equivalent. In degree $n \leqslant 3$, in fact, this is just enough for equivalence, which in turn proves LPP for the monomial case in these degrees. For $n = 2$, I proved LPP outright. Finally, I showed that given a Hilbert function $\mathcal{H}$, the LPP conjecture is true if and only if $\beta^{L_{\mathcal{H}}}_{i,\rho+i}$ is uniquely largest for all $i = 0, \ldots, n$ where $L_{\mathcal{H}}$ is the $\{a_1, \ldots, a_n\}$-lex plus powers ideal attaining $\mathcal{H}$ and ρ is the Castelnuovo-Mumford regularity of $R/L_{\mathcal{H}}$. Using the terminology of Macaulay 2, this says that it is enough to show that the last row of the lex plus powers Betti diagram is uniquely largest. The same argument shows that in order to prove the weaker conjecture, it is enough to show that the lex plus powers ideals have the largest number of generators in the last degree (among all ideals attaining the same Hilbert function and containing a regular sequence in the right degrees).

Future Plans: I hope to eventually show that the lex plus powers conjecture and the weaker conjecture are equivalent. Showing that the weaker conjecture implies that lex plus powers ideals have the largest socles was the first step in this regard, and I will try to extend the proof to give a more powerful result. To attack LPP in general, I plan to explore the feasibility of passing to a Gorenstein case, where self duality should prove useful. An obstruction to solving the conjecture has always been the inability to pass to the monomial case while preserving the regular sequence degrees. We can preserve degrees while passing to the Gorenstein case, however, and I am hopeful that some progress can be made using this fact.

Of course, questions concerning upper and lower bounds can be asked after fixing a Hilbert function and restricting to other special cases. Questions about the extremal resolutions of Gorenstein ideals have been considered by Geramita, Harima, and Shin [GHS] as well as Migliore and Nagel [MN], while Aramova, Hibi, and Herzog [AHH] have studied sharp upper bounds for the graded Betti numbers on the set of squarefree monomial ideals. I intend to work with both of these ideas. For Gorenstein ideals, I will attempt to establish a sharp upper bound in the cases left undone by previous work. I have already shown that in general there need not exist a sharp lower bound for the graded Betti numbers of Gorenstein ideals attaining a given Hilbert function [R1]. I will also attempt to show that sharp lower bounds need not exist when restricted to the set of squarefree monomial ideals attaining a given Hilbert function. It is worth noting that the squarefree monomial ideal case has broader implications. In particular, the conjecture is of interest to combinatorists, because squarefree monomial ideals can be described combinatorially with a unique simplicial complex.

The Buchsbaum, Eisenbud, Horrocks Conjecture: In 1977, Buchsbaum and Eisenbud conjectured that $\binom{n}{i}$ gives a global lower bound for the ith Betti number of any R-module M. This problem later appeared in Hartshorne's problem list [Ha] as a question contributed by G. Horrocks, and is sometimes referred to as Horrocks' problem. Although the conjecture is not difficult for $n \leqslant 4$, in general it has proven quite resistant. We know the following special cases: when M is the direct sum of R modulo monomial ideas [EG], when M is multigraded [S, Ch], when a free resolution of M has an associative multiplicative algebra structure [BE], when $M = R/I$ and I is in the linkage class of a complete intersection, and when $(x_1, \ldots, x_n)^2 M = 0$ [C]. Most recently, I showed (jointly with M. Hochster) that the bound holds if M can be embedded in the exact sequence $0 \to \oplus_{j=1}^{\xi}(R/I_j) \to M \to \oplus_{i=1}^{\psi}(R/J_i) \to 0$ where $I_1, \ldots, I_\xi$, and $J_1, \ldots, J_\psi \subset R$ are ideals generated by regular sequences, such that $J_i \subset I_j$ for all $i = 1, \ldots, \psi$ and $j = 1, \ldots, \xi$. This turns out to be an extension of the $(x_1, \ldots, x_n)^2 M = 0$ case. We can remove the hypothesis that $J_i \subset I_j$ if $|\psi - \xi| \gg 0$.

Future Plans: My work with M. Hochster on this problem is ongoing. We are currently considering how our ideas relate to certain moduli spaces of modules of finite length. We are hoping that a suitable conjecture about the components of these spaces may lead to progress on the Buchsbaum, Eisenbud, Horrocks conjecture. We furthermore expect the study of these components to be of independent interest.

We have also defined a certain interesting class of modules we call *general* modules. A general module M has the property that for no submodule $M' \subset M$ does there exists a $\hat{M}$ such that $0 \to M' \to \hat{M} \to M/M' \to 0$ is exact and the Betti numbers of $\hat{M}$ are smaller than those of M. Such modules exist, and are characterized by the property that for all $M' \subset M$, the ranks of the connecting homomorphisms in the long exact sequence in $\mathrm{Ext}(-, k)$ on the short exact sequence $0 \to M' \to M \to M/M' \to 0$ are maximal. We can pass from any given module to a general module that has Betti numbers no larger than the original, so it is enough to prove the conjecture for general modules. This motivates interest in the study of these modules, and I plan to explore them further.

There is a weaker form of the Buchsbaum-Eisenbud-Horrocks conjecture which claims that $\sum_{i=0}^{n} \alpha_i^M \geqslant 2^n$ for every finite length R-module M. This was shown by Dugger [Du] in the case that $M = R/I$ and I has $n+1$ generators, and by Stanley [St] whenever the length of M is odd, or more generally, writing[5] $\sum_{d=0}^{\infty} H(M,d)t^d = a_0 + a_1 t + a_2 t^2 + \cdots + a_p t^p$, if $\sum_i (-1)^i a_i \neq 0$. I plan to explore this conjecture for Gorenstein quotients of odd regularity (for such rings, $\sum_i (-1)^i a_i$ is identically zero). I expect the self duality of Gorenstein rings to prove helpful.

Generic Behavior: The final specific problem I would like to describe concerns calculating the graded Betti numbers, or the Hilbert series, of R/I when I is $(x_1, \ldots, x_n)$-primary in the generic case. This requires some notation. We say that a homogeneous form $f \in R_d$ is semi-regular if multiplication by f from $R_a \to R_{a+d}$ is either injective or surjective for all a. We say that a sequence of forms $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ is a semi-regular sequence if f_i is semi-regular on $R/(f_1, \ldots, f_{i-1})$ for all $i = 1, \ldots, r$. Given these conceptions, our goal is to understand the graded Betti numbers of R mod a semi-regular sequence. This is the generic case because the set of semi-regular sequences satisfies an open condition. In particular, if we consider the set of sequences $f_1, \ldots, f_r$ as an affine space where the coordinates give the coefficients of the polynomials, then the set of semi-regular sequences defines a Zariski open set.

Unfortunately, it is not known in general if this open set is nonempty. In fact, the existence of semi-regular sequences of degrees $d_1, \ldots, d_r$ is equivalent to an important conjecture of Fröberg. Given an infinite sum $\sum_i b_i t^i$ write $|\sum_i b_i t^i|$ to be the sum $\sum_i a_i t^i$ where $a_i = b_i$ until b_i is first negative, from which point we take $a_i = 0$. Then Fröberg has conjectured that the open set of forms $\{f_1, \ldots, f_r\}$ of degrees $d_1, \ldots, d_r$ which attain the Hilbert function

$$\left| \frac{\prod_{i=1}^{r}(1 - t^{d_i})}{(1-t)^n} \right|$$

is nonempty [Fr].

Fröberg's conjecture is known in a few cases. If $r \leqslant n$ then any regular sequence in degrees $d_1, \ldots, d_n$ has the Hilbert series required. The case $n \leqslant 3$ is due to Fröberg [Fr] and Anick [A]. Stanley has proved the case where $r = n+1$ and $\mathrm{char}(k) = 0$. Hochster and Laksov [HL] have shown that if $d_1 = \cdots = d_r = d$, then the conjectured formula predicts the right value in degree $d+1$. Aubry [Au] has extended their work to certain other values. Finally, Fröberg and Hollman [FH] have established several cases using computer experimentation.

Currently, I am engaged in joint work with Keith Pardue to explore the resolutions of semi-regular sequences. Our first paper, which is in the final stages of composition, includes several important results [PR]. Let ρ be the Castelnuovo-Mumford regularity of R/I for I generated by a semi-regular sequence $f_1, \ldots, f_r$. Our main theorem shows that a minimal

[5] It is known that this infinite sum is a polynomial when M is finite length

free resolution of R/I is isomorphic to the Koszul complex on the f_i in degrees $\leqslant \rho - 2$. If the coefficient of $t^{\rho+1}$ in the series

$$\frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^n}$$

is zero, where we have written d_i to be the degree of f_i, then we can show that a minimal free resolution of R/I is isomorphic to the Koszul complex on the f_i in degrees $\leqslant \rho - 1$ (when the coefficient of t^ρ is zero, we say that the d_i satisfy the *special numerical condition*). An easy corollary allows us to compute the complete resolution of R/I in this case. Note that we obtain more then just the graded Betti numbers. The theorem also calculates the differentials of a resolution of R/I for degree $\leqslant \rho-2$ or $\leqslant \rho-1$. We should mention that the special numerical condition is not as strange as it first seems. If $r = n+1$, then the condition holds if and only if $\sum_{i=1}^{n+1}(d_i - 1)$ is odd. Recall that semi-regular sequences are known to exist for $r = n + 1$ if the characteristic of k is zero. Thus in characteristic zero, our theorem computes the graded Betti numbers of generic almost complete intersections exactly half of the time.

We can also prove a partial converse to the main theorem. Let $I = (f_1, \ldots, f_r)$ be any $(x_1, \ldots, x_n)$-primary ideal, and suppose that the first term (counting from zero, of course) in a free resolution of R/I is isomorphic to the first term in the Koszul complex on the f_i in degrees $\leqslant \rho - 2$. Then I can be generated by a semi-regular sequence $g_1, \ldots, g_t$, and a minimal free resolution of R/I is isomorphic to the Koszul complex on the g_i in degrees $\leqslant \rho - 2$. We are not yet able to determine if I can be minimally generated by a semi-regular sequence in this situation.

We should mention that this work has similarities with work of Juan Migliore and Rosa Miró-Roig, as well as Karen Chandler. In a recently accepted paper, Migliore and Miró-Roig obtained a result similar to our main theorem, though different in scope and with a different proof [MM2]. In an earlier paper they did important work on the $r = n + 1$ case [MM1].

Future Plans: Ultimately, of course, we would like to calculate the graded Betti numbers for all semi-regular sequences. There are several approaches to take. For five forms in three variables (the smallest unknown case) our theorem only gives a complete answer when the degrees of the forms satisfy the special numerical condition. When the ideal $J = (f_1, f_2, f_3, f_4) : (f_5)$ is itself generic, however, it appears likely that our main theorem, the 4-generated case, and the mapping cone will be sufficient to describe the graded Betti numbers of $(f_1, f_2, f_3, f_4, f_5)$. It may be helpful consider the case where all forms have the same degree.

We are also very interested in the following statement which we refer to as the *Generic Socle Hypothesis*: the socle in degree d for a generic ideal generated by r forms in n variables is $\max\{0, H(S/I, d) - nH(S/I, d+1)\}$. We know, in fact, that this hypothesis is not always true; examples are easy to find if $\dim(R) = 3$. After calculating a large number of examples, however, it seems that the hypothesis is predominately true in more than three variables. We suspect that understanding when and why the Generic Socle Hypothesis fails will eventually help us compute the graded Betti numbers which our theorem misses.

Calculating the graded Betti numbers of a semi-regular sequence is interesting both because the resolution of the generic ideal is minimal, and because understanding such resolutions may shed light on Fröberg's conjecture. I hope to make a direct assault on Fröberg's conjecture as well.

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