1. (p.85 #23) $\mathbb{Z}_2$

2. (p.85 #24) $V$ (the Klein 4-group)

3. (p.85 #26) $\mathbb{Z}_2$

4. $h(1)(n) = 1+n$, so $h(1)(1) = 2$, $h(1)(2) = 3$, $h(1)(3) = 4$
   $h(1)(4) = 0$ and $h(1)(0) = 1$

   $h(3)(n) = 3+n$, so $h(3)(0) = 3$, $h(3)(1) = 4$, $h(3)(2) = 0$
   $h(3)(3) = 1$, $h(3)(4) = 2$

5. (a) Since $ea = a = ae$, $e \in C(e)$ and so $C(e) \neq \emptyset$.
   Now let $g, h \in C(e)$. Then $(gh)a = g(ha) = g(ae)$
   $= (ga)e$
   $= (ag)e$
   $= e(ge)$

   And so $gh \in C(e)$

   Also note that $g^{-1}a = g^{-1}a(gg^{-1})$
   $= g^{-1}agg^{-1}$
   $= g^{-1}(ga)g^{-1}$
   $= (g^{-1}g)ag^{-1} = ag^{-1}$

   and so $g^{-1} \in C(e)$. 
   By the subgroup test, $C(e) < G$. 

6. (p. 96 #12) \( V = (1, 3, 4, 7, 8, 6, 5, 2) \)

7. (p. 96 #21) Let \( H = H \cap A_n \). If \( H = H_e \), then every element of \( H \) is even. Otherwise, \( H \) is a proper subgroup of \( H \). In this case, pick \( t \in H \setminus H_e \).

Claim: \( H \setminus H_e = \{ H_e, t \} \)

Proof: Let \( v \in H \). If \( v \notin H_e \), then \( vH_e = H_e \). It is odd, which implies that \( v^{-1} \notin H_e \).

It follows that \( vH_e = H_e \).

Claim: Done.

Since \( H = H_e \cup H \), \( H_e \cap H = \emptyset \) and \( |H_e| = |H| \)

We conclude that \( |H_e| = 1 \cdot |H_e| = |H| \)

8. (p. 96 #31) For each \( v \in S_n \), let \( M_v = \{ v(a) \mid v(a) \neq a \} \).

Then, \( H \) consists of all \( v \in S_n \) such that \( |M_v| < \infty \).

First note that \( M_v = \{ \} \) (where \( i \) denotes the identity permutation) and so \( i \in H \).

Let \( \sigma, \tau \in H \). If \( \sigma \tau(a) \neq a \), then \( a \in M_\sigma \cup M_\tau \)

(\( a \) must be moved by \( \sigma \) or \( \tau \) or both) and so \( M_\sigma \cap M_\tau \neq \emptyset \).

Since \( |M_\sigma \cup M_\tau| < \infty \) we conclude that \( |M_\sigma| < \infty \) and \( |M_\tau| < \infty \).

Also, \( \sigma^{-1}(a) = a \) if and only if \( \sigma^{-1}(a) \neq a \), which implies that \( M_{\sigma^{-1}} = M_\sigma \) is finite. \( \therefore \sigma^{-1} \in H \).

By the subgroup test, \( H \leq S_n \).
9. \( (p, 101 + 6) \quad H = \mathbb{Z}(p, \mu_2) \quad \mathbb{C}, \quad H = \mathbb{Z}(p, \delta_2) \),
\( C_2 = \mathbb{Z}(p, \mu_2) \quad \mathbb{C}^2 \), \( C_3 = \mathbb{Z}(p, \delta_2) \)

is a complete list of the Sylow classes of \( D_{4} / H \).

10. By direct computation, \( |\mu| = 4 \) and so
\( (S_4, <\mu>) = \frac{6!}{4} = 6 \cdot 5 \cdot 3 \cdot 2 \cdot 1 = 180 \)

11. Let \( x \in gH \). Then \( x = gh \) for some \( h \in H \).
Now \( x^{-1} = gh^{-1} \in H \) and so \( xg^{-1} = h \) for some \( h \in H \). It follows that \( x = hg \in Hg \).

\[ x \in gH \] The proof that \( Hg \subseteq gH \) is similar.

12. (a) Since \( (1 + 2i)^{-1} (2 + i) = \frac{4 + 3i}{5} \) and
\[ 1 \frac{4 + 3i}{5} = 1 \] are candidates that \( (1 + 2i)u = (2 + i)u \).

(b) Since \( \frac{4 + 3i}{5} \) is \( (1 + 2i)u \neq 3u \).

(c) Since \( |a^{-1}b| = 1 \) if and only if \( |ab| = 16 \) it follows that all \( 6u \) if and only if \( |a| = 16 \). Thus, the set \( 6u \) is a circle in the complex plane consisting of all complex numbers of modulus \( 16 \).

13. Write \( |g| = m \). Since \( m \mid n \), there exists \( t \in \mathbb{Z} \) such that \( n = tm \). Now \( g^m = g^{tm} = (g^t)^m = e^t = e \).
7 (p. 96 #28) Alternate proof

Let $H_0 = H_0 \cap A_n$. If $H \subseteq H_0$, then every element of $H$ is even. Otherwise, choose $x \in H - H_0$. Since $x$ is odd, every element of $H_0 = \{ x \in H \mid x \in H_0 \}$ is also odd.

Now observe that the mapping $f : H_0 \to H_0$ given by $f(x) = x^2$ is a bijection and so $|H_0| = |H_0|$. Claim: $H - H_0 = H_0$.

Proof: The inclusion $H_0 \subseteq H - H_0$ is obvious.

Let $x \in H - H_0$ and note that $x^2 \in H_0$.

It follows that $x = (x^2)^{1/2} = (x^{-2})^2 \in H_0$

and so $H - H_0 \subseteq H_0$.

Claim Done.

Now we have $H = H_0 \cup H_0$, where $H_0 \cap H_0 = \emptyset$ and $|H_0| = |H_0|$ and so $|H_0| = |H|/2$.

That is $H$ consists of the same number of even and odd permutations.