

## AN ABSTRACT OF THE DISSERTATION OF

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Abstract approved: \_\_\_\_\_

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A Coxeter group  $W$  is said to be *rigid* if, given any two Coxeter systems  $(W, S)$  and  $(W, S')$ , there is an automorphism  $\rho$  of  $W$  which carries  $S$  to  $S'$ . In this dissertation we review rigidity results of D. Radcliffe [29] and of R. Charney and M. Davis [11], noting that certain restrictions must be placed on the Coxeter group in order to obtain rigidity. Radcliffe examined “right-angled” Coxeter groups, while Charney and Davis considered Coxeter groups of “type  $HM_n$ ” (both of which are defined in Section 4.2).

We introduce Coxeter groups of “type  $K_n$ ,” which may be viewed as a natural generalization of type  $HM_2$  Coxeter groups. Relying primarily on techniques of geometric group theory, we show that if  $W$  is Coxeter group of type  $K_n$  with Coxeter systems  $(W, S)$  and  $(W, S')$ , then the associated Coxeter graphs  $\Gamma_S$  and  $\Gamma_{S'}$  are isomorphic. With added restrictions on the system  $(W, S)$  this is sufficient to conclude that  $W$  is rigid.

The geometric analysis mentioned above is performed on a simplicial complex  $\Sigma$ , called the the Davis complex, which was introduced by M. Davis in [13]. G. Moussong [24] showed that the Davis complex  $\Sigma$  supports a  $CAT(0)$  metric  $d_M$  (called the Moussong metric) and that the Coxeter group  $W$  acts properly

and cocompactly by isometries on the metric space  $(\Sigma, d_M)$ . This geometric structure enables one to deduce a wealth of information about the Coxeter group.

We observe that passing from type  $HM_2$  Coxeter groups to type  $K_n$  Coxeter groups introduces greater topological complexity in the Davis complex  $\Sigma$ . We undertake a study of certain subcomplexes of  $\Sigma$  with the goal of alleviating the complications. In doing so we obtain a topological proof that if  $r \in W$  is a reflection and  $W$  is of type  $K_n$ , then the centralizer  $C_W(r)$  is isomorphic to  $\mathbb{Z}_2 \times F$ , where  $F$  is a free group. For general Coxeter groups, a similar result has been obtained by B. Brink [9] using different methods.

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by

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I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of dissertation to any reader upon request.

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Anton Kaul, Author

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# RIGIDITY FOR A CLASS OF COXETER GROUPS

## 1 Introduction

### 1.1 Statement of the Problem

A *Coxeter system* is a triple  $(W, S, m)$  where  $W$  is a group,  $S \subseteq W$ , and

$$m : S \times S \longrightarrow \mathbb{N} \cup \{\infty\}$$

such that

1. for all  $s, t \in S$ ,  $m(s, t) = 1$  if and only if  $s = t$ ;
2.  $m(s, t) = m(t, s)$  for all  $s, t \in S$ ;
3.  $W$  has a presentation of the form

$$\langle S : (st)^{m(s,t)}; s, t \in S \rangle .$$

$W$  is called a *Coxeter group* (relations of the form  $(st)^\infty$  are, of course, discarded). In this dissertation we consider the following question: given any Coxeter systems  $(W, S)$  and  $(W, S')$  for  $W$ , is there an automorphism  $\rho : W \longrightarrow W$  such that  $\rho(S) = S'$ ? If the answer is in the affirmative,  $W$  is said to be *rigid*. It is known that not all Coxeter groups are rigid, but known examples of such groups are rare. However, in the presence of certain restrictions on  $(W, S)$ , rigidity for  $W$  can be attained.

## 1.2 Organizational Overview

We outline the organization of the dissertation. In Chapter 2 we introduce various group theoretical prerequisites that are necessary to proceed. In particular we describe relatively recent developments in geometric group theory, due primarily to M. Gromov (i.e., the use of non-positive curvature in the study of infinite groups), which are essential in our approach to the rigidity question.

In Chapter 3 we detail the standard terminology and results on reflection groups and Coxeter groups. Classically, Coxeter groups were studied via their “root systems.” Roughly speaking, this approach involves viewing a Coxeter group  $W$  with system  $(W, S)$  as a group of real matrices such that each element  $s \in S$  (and its  $W$ -conjugates) acts on a real vector space  $V$  by reflection across a hyperplane  $H$  of codimension 1 in  $V$ , and interchanges the components of  $V - H$  (cf. Theorems 3.3.1 and 3.3.2). More recently it has been shown by M. Davis [13] that given a Coxeter system  $(W, S)$ , the group  $W$  acts on a contractible simplicial complex  $\Sigma(S)$  by reflections (see Theorem 3.4.2). The complex  $\Sigma(S)$  is called the *Davis complex* associated to the system  $(W, S)$ . A result of G. Moussong [24] (stated as Theorem 3.4.3) provides the integral link between the theory of Coxeter groups and non-positive curvature.

There are surprisingly few known results concerning the rigidity problem for Coxeter groups. Chapter 4 details results of D. Radcliffe [29] and of R. Charney and M. Davis [11] concerning rigidity. In each case, a particular class of Coxeter groups is isolated and shown to be rigid. Radcliffe showed that “right-angled” Coxeter groups are rigid (defined and stated as Theorem 4.2.1). Charney and Davis proved that Coxeter groups of “type  $HM_n$ ” are rigid (defined and stated as Theorem 4.2.2).

In Chapter 5 we introduce Coxeter groups of “type  $K_n$ .” This class of groups can be viewed as a natural generalization of the type  $HM_2$  groups of Charney and Davis. Type  $K_n$  Coxeter groups are obtained by relaxing the homological restrictions imposed on  $HM_2$  groups. The reader should be warned that type  $HM_2$  Coxeter groups are *not* type  $K_n$  Coxeter groups in general (the definition of type  $K_n$  Coxeter groups is given in Section 5.1). It will be shown that if  $W$  is a Coxeter group of type  $K_n$  with Coxeter systems  $(W, S)$  and  $(W, S')$ , then the Coxeter graphs  $\Gamma_S$  and  $\Gamma_{S'}$  are isomorphic (Corollary 5.4.7). In certain cases this is sufficient to conclude that  $W$  is rigid (Corollary 5.4.8).

As will be seen, the price paid in passing from the type  $HM_2$  groups to the type  $K_n$  groups is an increase in the complexity of the topology of the Davis complex. We take note of these complications and in Chapter 6 take steps to unravel the complexity.

## 2 Group Theoretical Prerequisites

### 2.1 Combinatorial Group Theory

Given any set  $X$  there is a group  $F = F(X)$  such that  $X \subseteq F$  satisfies the following: for any group  $H$  and any function  $f : X \rightarrow H$ , there is a unique homomorphism  $g : F \rightarrow H$  such that  $g|_X = f$  (where  $g|_X$  is the restriction of  $g$  to  $X$ ). The group  $F$  is called a *free group* with *basis*  $X$ . For the details on the construction of free groups see [23], Chapter 1.

Let  $R \subseteq F$  be any subset of of the free group  $F$ . The *normal closure* of  $R$ , denoted  $\langle\langle R \rangle\rangle$ , is the smallest normal subgroup of  $F$  containing  $R$ . A group  $G$  has *presentation*  $\mathcal{P} = \langle X : R \rangle$  if  $G$  is isomorphic to the quotient  $F/\langle\langle R \rangle\rangle$ . Elements of  $X$  are called *generators* and elements of  $R$  are called *relators*. If  $\{x_1, x_2, \dots, x_k\} \subseteq X$ , the product  $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}$  (where  $\varepsilon_i = \pm 1$  for  $1 \leq i \leq k$ ) is called a *word* in the generators of  $\mathcal{P}$ .

A group  $G$  is *finitely generated* (resp. *finitely related*) if it has a presentation  $\mathcal{P} = \langle X : R \rangle$  with  $X$  (resp.  $R$ ) a finite set. A group  $G$  is *finitely presented* if it has a presentation  $\mathcal{P} = \langle X : R \rangle$  with both  $X$  and  $R$  finite.

Combinatorial group theory is the study of groups defined by presentations. Historically, the field of combinatorial group theory can trace its origins to a paper of von Dyke [16] in 1882, in which he defined discrete groups of isometries of hyperbolic space in terms of generators and relations. The major impetus to study groups defined by presentations was provided by Max Dehn in the early twentieth century. In the study of low-dimensional manifolds, the fundamental group is of major importance. Dehn was working on classifying such manifolds (specifically surfaces and knot-complements) and, when given a specific manifold, the fundamental group often arises in the form of a presentation. In this

way he came to recognize the deep connection between group theory and topology as well as the realization that the study of combinatorial group theory was of fundamental importance.

With this idea in mind, in 1912 Dehn [15] articulated three basic problems which arise naturally from topological considerations and which concern group theorists to this day.

1. *Word Problem:* Given a presentation  $\mathcal{P}$  for a group  $G$ , is there an algorithm which, for any word  $w$  in the generators of  $\mathcal{P}$ , will determine in a finite number of steps whether  $w$  represents the trivial element of  $G$ ? If such an algorithm exists, then  $\mathcal{P}$  has *solvable word problem*. If  $\mathcal{P}$  and  $\mathcal{Q}$  are finite presentations for a group  $G$ , it can be shown that  $\mathcal{P}$  has solvable word problem if and only if  $\mathcal{Q}$  has solvable word problem. Thus we may speak of a finitely presented group  $G$  as having solvable word problem.
2. *Conjugacy Problem:* Given a presentation  $\mathcal{P}$  for a group  $G$ , is there an algorithm which, for any words  $u$  and  $v$  in the generators for  $\mathcal{P}$ , will determine in a finite number of steps whether  $u$  and  $v$  represent elements of  $G$  which are conjugate in  $G$  (i.e.,  $u = wvw^{-1}$  for some  $w \in G$ )? If such an algorithm exists, then  $\mathcal{P}$  has *solvable conjugacy problem*. Note that if  $\mathcal{P}$  has solvable conjugacy problem, then it also has solvable word problem.
3. *Isomorphism Problem:* Given presentations  $\mathcal{P}$  and  $\mathcal{Q}$  is there an algorithm which will determine in a finite number of steps whether the groups determined by the presentations are isomorphic?

It has been shown that in general, each of these problems is unsolvable. In particular, Boone [5] and Novikov [26], [27] showed that there exists a finitely presented group with unsolvable word problem. There are however, many classes

of groups for which one or more of these problems is solvable. For example, free groups have solvable word problem. It will be seen (Theorem 3.4.4) that Coxeter groups have solvable conjugacy problem (this result is a consequence of a theorem of Moussong [24]).

Suppose  $S_1$  and  $S_2$  are orientable surfaces and  $\mathcal{P}$  and  $\mathcal{Q}$  are presentations for the fundamental groups  $\pi_1(S_1)$  and  $\pi_1(S_2)$ , respectively. It is known that any orientable surface  $S$  is determined up to homeomorphism by the homology group  $H_1(S)$ , and that  $H_1(S)$  is isomorphic to the abelianization of the fundamental group  $G = \pi_1(S)$  (i.e.,  $H_1(S) \cong G/[G, G]$ , where  $[G, G]$  denotes the commutator subgroup of  $G$ ; see [17], Chapter 12 for the details). It is also known that if  $g$  is the genus of the surface  $S$ , then

$$H_1(S) \cong \begin{cases} 0 & \text{if } g = 0 \\ \mathbb{Z}^{2g} & \text{if } g > 0 \end{cases} .$$

Thus, to determine whether  $\mathcal{P}$  and  $\mathcal{Q}$  present isomorphic groups, it is sufficient to determine whether the homology groups  $H_1(S_1)$  and  $H_1(S_2)$  are isomorphic and, with presentations  $\mathcal{P}$  and  $\mathcal{Q}$  in hand, this is an easy task. Consequently, the isomorphism problem is solvable for presentations of the fundamental groups of orientable surfaces (in fact, the hypothesis of orientability is unnecessary). For the details on the homology groups of surfaces, see [17], Chapter 19.

The reader should observe that the rigidity question for Coxeter groups is a special case of the isomorphism problem in the following sense. Suppose one is given Coxeter systems  $(W, S, m)$  and  $(W', S', m')$  which determine presentations

$$\mathcal{P}_S = \langle S : m(s, t); \text{ for all } s, t \in S \rangle$$

and

$$\mathcal{P}_{S'} = \langle S' : m'(s', t'); \text{ for all } s', t' \in S' \rangle .$$

The object of rigidity is to determine whether the presentations  $\mathcal{P}_S$  and  $\mathcal{P}_{S'}$  determine isomorphic groups *and* if so, is there an isomorphism between the groups determined by the presentations which carries  $S$  to  $S'$ ?

## 2.2 Group Actions

Given a group  $G$  and a set  $X$ , a *left action* of  $G$  on  $X$  is a function

$$f : G \times X \longrightarrow X$$

satisfying

1.  $f(1, x) = x$  for all  $x \in X$ ;
2. if  $g, h \in G$ , then  $f(gh, x) = f(g, f(h, x))$  for all  $x \in X$ .

There is an analogous definition for a *right action* of  $G$  on  $X$ . All group actions in this paper will be left actions and will be referred to as  $G$ -actions. If  $g \in G$ ,  $x \in X$ , and  $f$  is an action of  $G$  on  $X$ , we will denote  $f(g, x)$  by  $g \cdot x$  or simply  $gx$ . A  $G$ -action on  $X$  is said to be *free* if, for all  $x \in X$ ,  $gx = x$  if and only if  $g = 1$ . Given a subset  $Y \subseteq X$ , the (setwise) *stabilizer* of  $Y$  is

$$\text{stab}(Y) = \{g \in G : g \cdot Y = Y\},$$

which is the set of all  $g \in G$  which leave the set  $Y$  invariant. It is easy to show that  $\text{stab}(Y)$  is a subgroup of  $G$ . If  $y \in X$ , the *orbit* of  $y$  in  $X$  is the set

$$G \cdot y = \{g \cdot y : g \in G, y \in Y\}.$$

The  $G$ -action is *transitive* on  $X$  if, for any  $x, y \in X$ , there is a  $g \in G$  such that  $gx = y$ .

Suppose  $X$  is a topological space and  $G$  is a group acting on  $X$ . The space  $X$  is a (left)  $G$ -space if, for each  $g \in G$ , the map  $\tau_g : X \rightarrow X$  given by  $\tau_g(x) = gx$  for all  $x \in X$  is continuous. If this is the case it is easy to show that for each  $g \in G$ , the map  $\tau_g$  is in fact a homeomorphism.

A metric space  $(X, d)$  is said to be *proper* if every closed, bounded subset is compact. For the remainder of this section, we assume  $(X, d)$  is a proper metric space with  $G$  acting on  $X$  by isometries (i.e., for every  $g \in G$ ,  $d(x, y) = d(gx, gy)$  for all  $x, y \in X$ ). The action of  $G$  on  $X$  is said to be *proper* if for every bounded set  $B \subseteq X$ , the set

$$\{g \in G : gB \cap B \neq \emptyset\}$$

is finite. Note that for a proper action, the stabilizer of each point  $x \in X$  is finite. The *orbit space* is the quotient

$$G \backslash X = \{G \cdot x : x \in X\},$$

topologized with the quotient topology. We define a distance function

$$d_G : G \backslash X \times G \backslash X \rightarrow \mathbb{R}$$

on  $G \backslash X$  by

$$d_G(Gx, Gy) = \inf\{d(a, b) : a \in Gx, b \in Gy\}.$$

If  $G$  acts properly on  $X$  it can be shown that  $d_G$  is a metric on  $G \backslash X$ ; in such a case the metric topology is equivalent to the quotient topology on  $G \backslash X$  (see eg. [30], §6.5). The action of  $G$  on  $X$  is *cocompact* if  $G \backslash X$  is compact.

## 2.3 Geometric Group Theory

Geometric group theory is the study of the interplay between groups and geometry/topology. The approach to this theory comes in two flavors. The first

is to treat a given group  $G$  as a geometric object. One such way to do this is by putting a metric on  $G$ . For example, if  $G$  has a presentation  $\mathcal{P} = \langle X : R \rangle$ , the *word metric* on  $G$  with respect to the presentation  $\mathcal{P}$  is defined as follows: for  $g, h \in G$  let

$$d(g, h) = \inf\{n : g^{-1}h = x_1^{\varepsilon_1}x_2^{\varepsilon_2}\cdots x_n^{\varepsilon_n}\}$$

where  $x_1, x_2, \dots, x_n \in X$  and  $\varepsilon_i = \pm 1$  for  $1 \leq i \leq n$ . In other words, the distance between  $g$  and  $h$  is the minimal number of letters required to form a word which represents  $g^{-1}h$  in  $G$ . There are many other ways to metrize a group (see eg. [30], §5). The second is to locate a topological space  $X$ , (preferably a metric space) on which  $G$  acts (preferably by isometries). Studying the  $G$ -action on  $X$  yields information on the structure of  $G$ . In both of these approaches, the notion of curvature (to be defined below) plays an important role.

Dehn laid the foundation for a geometric approach to combinatorial group theory when he established the relationship between low-dimensional topology and group theory. For example, orientable surfaces of positive genus support metrics of non-positive curvature.

Though the connections between the combinatorial and geometric aspects of group theory are ubiquitous, the role of geometry was largely ignored during most of the development of the combinatorial theory. Its emergence as a central player in group theory is due largely to the work of M. Gromov [18], [19], [20], in the latter part of the twentieth century.

## 2.4 Geometric Notions

A *geodesic space* is a metric space  $(X, d)$  such that for any two points  $x, y \in X$ , there exist  $a, b \in \mathbb{R}$  and an isometric embedding  $\gamma : [a, b] \rightarrow X$

such that  $\gamma(a) = x$  and  $\gamma(b) = y$ . The map  $\gamma$  is called a *geodesic arc*. Denote the image  $\gamma([a, b])$  in  $X$  by  $[x, y]$ . The set  $[x, y]$  is called a *geodesic segment* connecting  $x$  and  $y$ . A *geodesic line* is a local isometry  $\alpha : \mathbb{R} \rightarrow X$  of the real line into  $X$  (i.e., for every  $x \in \mathbb{R}$  there is an open interval  $I \subseteq \mathbb{R}$  such that the restriction of  $\alpha$  to  $I$  is an isometric embedding). The image  $\alpha(\mathbb{R})$  in  $X$  is called a *geodesic*.

If  $\gamma : [a, b] \rightarrow X$  is a geodesic arc, we consider the path  $\bar{\gamma} : [0, 1] \rightarrow X$  defined by

$$\bar{\gamma}(t) = \gamma((1-t)a + tb)$$

for all  $t \in [0, 1]$ . Notice that the image of  $\bar{\gamma}$  is equal (setwise) to the image of  $\gamma$ . The path  $\bar{\gamma}$  is a *parameterization of  $\gamma$  proportional to arc length*.

Given a geodesic space  $(X, d)$ , the metric is said to be *convex* if for each pair of geodesic arcs

$$\gamma_1 : [0, 1] \rightarrow X \text{ and } \gamma_2 : [0, 1] \rightarrow X,$$

parametrized proportional to arc length, the inequality

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1))$$

is satisfied for all  $t \in [0, 1]$ . It is immediately evident that geodesic segments are unique if the metric is convex.

Geodesic segments are not in general unique. For example, the unit sphere  $S^2$  with the standard metric is a geodesic space. The geodesics of  $S^2$  correspond to “great circles” (i.e., closed arcs in  $S^2$  which are isometric to the equator). It can be shown that if  $x, y \in S^2$  are antipodal points, then there are infinitely many geodesic segments connecting  $x$  and  $y$  (if  $x$  and  $y$  are the north and south poles respectively, each longitude is a geodesic segment connecting  $x$  and  $y$ ; see [30], §2.1 for the details).

The following geodesic spaces are used as standard models in geometry. Let  $\chi \in \mathbb{R}$  be given, and let  $\mathbb{M}_\chi$  denote either the 2-sphere of radius  $1/\sqrt{\chi}$ , the Euclidean plane, or 2-dimensional hyperbolic space of curvature  $\chi$ , according to  $\chi > 0$ ,  $\chi = 0$ , or  $\chi < 0$ , respectively. Let  $d_\chi$  denote the standard metric on  $\mathbb{M}_\chi$  (see [30], Chapters 1-3 for the details).

If  $x, y, z \in X$ , a *triangle*  $(x, y, z)$  consists of a choice of geodesic segments  $[x, y]$ ,  $[y, z]$ , and  $[z, x]$ . A *comparison triangle* for  $(x, y, z)$  is a triangle  $(x', y', z')$  in  $\mathbb{M}_\chi$  such that

$$d(x, y) = d_\chi(x', y'), \quad d(y, z) = d_\chi(y', z'), \quad \text{and} \quad d(z, x) = d_\chi(z', x').$$

A geodesic space  $(X, d)$  is a *CAT*( $\chi$ ) *space* if, for any triangle  $(x, y, z)$  in  $X$ , and any  $p \in [y, z]$ , the *CAT*( $\chi$ ) *inequality*

$$d(p, x) \leq d_\chi(p', x')$$

is satisfied, where  $p'$  is the image of  $p$  under the isometry  $[y, z] \longrightarrow [y', z']$ . A geodesic space  $X$  has *curvature*  $K_X \leq \chi$  if every point  $p \in X$  has a neighborhood  $U_p$  such that every triangle in  $U_p$  satisfies the *CAT*( $\chi$ ) inequality. The theory of *CAT*( $\chi$ ) spaces was developed by the Russian school of A. D. Aleksandrov

(*C* = comparison, *A* = Aleksandrov, *T* = Toponogov)

to encompass the theory of Riemannian manifolds and extend this theory to a more general setting. For more on *CAT*( $\chi$ ) spaces see [1], [2], [4], or [8].

Some important facts about *CAT*(0) spaces are contained in the following results.

**Theorem 2.4.1** *If  $(X, d)$  is a CAT(0) space, then the metric  $d$  is convex.*

Proof: As a special case, let  $\alpha, \beta : [0, 1] \rightarrow X$  be geodesics with  $\alpha(0) = \beta(0)$  and consider a triangle  $(\alpha(0), \alpha(1), \beta(1))$  in  $X$  and the comparison triangle  $(\alpha(0)', \alpha(1)', \beta(1)')$  in  $\mathbb{R}^2$  with the Euclidean metric  $d_R$  such that

$$\alpha(0)' = \beta(0)' = (0, 0) \in \mathbb{R}^2.$$

If  $t \in [0, 1]$ , then

$$d_R(\alpha(t)', \beta(t)') = td_R(\alpha(1)', \beta(1)') = td(\alpha(1), \beta(1)).$$

By the CAT(0) inequality,

$$d(\alpha(t), \beta(t)) \leq d_R(\alpha(t)', \beta(t)')$$

which implies that

$$d(\alpha(t), \beta(t)) \leq td(\alpha(1), \beta(1))$$

for all  $t \in [0, 1]$ .

In the general case, we let  $\alpha, \beta : [0, 1] \rightarrow X$  be (reparameterized) geodesics and let  $\gamma : [0, 1] \rightarrow X$  be the reparameterized geodesic with  $\gamma(0) = \alpha(0)$  and  $\gamma(1) = \beta(1)$ . By applying the special case we obtain

$$d(\alpha(t), \gamma(t)) \leq td(\alpha(1), \gamma(1))$$

and

$$d(\gamma(t), \beta(t)) \leq (1 - t)d(\gamma(0), \beta(0))$$

for all  $t \in [0, 1]$ . This implies that

$$\begin{aligned}
d(\alpha(t), \beta(t)) &\leq d(\alpha(t), \gamma(t)) + d(\gamma(t), \beta(t)) \\
&\leq td(\alpha(1), \beta(1)) + (1-t)d(\alpha(0), \beta(0))
\end{aligned}$$

for all  $t \in [0, 1]$  as desired.  $\square$

Theorem 2.4.1 is particularly useful because geodesic segments are unique in a space with a convex metric. In addition, the convexity of the metric in a  $CAT(0)$  space implies that geodesic segments vary continuously with the endpoints. As a consequence we have:

**Corollary 2.4.2** *If  $(X, d)$  is a  $CAT(0)$  space, then  $X$  is contractible.*

Proof: Fix  $x_0 \in X$ . For each  $x \in X$ , let  $\alpha_x : [0, 1] \rightarrow X$  be the reparameterized geodesic from  $x_0$  to  $x$ . Define a homotopy

$$H : X \times [0, 1] \rightarrow X$$

by  $H(x, t) = \alpha_x(t)$ . Applying Theorem 2.4.1, we see that  $H$  is continuous and so  $X$  is contractible.  $\square$

If  $X$  is a  $CAT(\chi)$  space, and a group  $G$  acts properly and cocompactly by isometries on  $X$ , then  $G$  is called a  $CAT(\chi)$  group. As previously mentioned, the role of curvature plays a central role in geometric group theory. This is especially true in the case of non-positive curvature (i.e.  $CAT(0)$  groups). The following theorems illustrate this fact.

**Theorem 2.4.3** ([7], Corollary 2.7) *If  $G$  is a finite group acting by isometries on a complete  $CAT(0)$  space  $X$ , then there is an  $x \in X$  such that  $gx = x$  for all  $g \in G$ .*

**Theorem 2.4.4** ([8], Corollary II.2.8) *If  $G$  is a finite group acting by isometries on a complete  $CAT(0)$  space, then the set  $C$  of points of  $X$  fixed by  $G$  is a convex subspace of  $X$  (hence  $C$  is contractible).*

**Theorem 2.4.5** ([8], Theorem III.Γ.1.12) *If  $G$  is a  $CAT(0)$  group, then  $G$  has solvable conjugacy problem (and therefore solvable word problem).*

For a comprehensive study of the theory of  $CAT(0)$  groups, the reader is referred to [8].

### 3 Coxeter Groups and Reflection Groups

#### 3.1 Reflection Groups

In this section we collect some standard results on the theory of reflection groups. Our approach follows [6]. For alternate approaches see [10] or [22].

If  $X$  is a topological space and  $A \subseteq X$  is a subspace of  $X$ , we denote the closure of  $A$  in  $X$  by  $\overline{A}$  and we denote the interior of the subspace  $A$  by  $\text{int}A$ . Let  $X$  be a connected, locally path connected, Hausdorff space. A *reflection* on  $X$  is a homeomorphism  $r : X \rightarrow X$  such that

- A1.  $r^2 = 1_X$  (where  $1_X$  denotes the identity map on  $X$ );
- A2. the fixed-point set  $M_r = \{x \in X : r(x) = x\}$  separates  $X$  into two nonempty components and these components are interchanged by the reflection  $r$ ;
- A3. every  $x \in M_r$  has an arbitrarily small connected open neighborhood  $U_x$  which is separated into two components by  $M_r$  and these components are interchanged by the reflection  $r$ .

The set  $M_r$  is called the *wall* or *mirror* of the reflection  $r$ . Note that since  $X$  is Hausdorff, for every reflection  $r$  of  $X$ , the mirror  $M_r$  is a closed subset of  $X$ .

A group  $\Gamma$  of homeomorphisms of  $X$  is called a *reflection group* if

- R1.  $\Gamma$  is generated by reflections;
- R2. the collection of mirrors  $\{M_r : r \text{ is a reflection in } \Gamma\}$  is a locally finite family in  $X$  (i.e., each point  $z \in X$  has an open neighborhood  $U_z$  such that the intersection  $M_r \cap U_z$  is nonempty for at most finitely many reflections  $r \in \Gamma$ );

R3. if  $r$  and  $r'$  are distinct reflections in  $\Gamma$ , then any path  $\alpha : [0, 1] \longrightarrow X$  can be approximated arbitrarily closely by a path which misses the intersection  $M_r \cap M_{r'}$  (where  $\alpha$  is viewed in  $\mathcal{C}([0, 1], X)$ , the space of continuous functions from the interval  $[0, 1]$  into  $X$  with the compact-open topology).

Common examples of reflection groups are the dihedral group  $D_n$  (with  $2n$  elements) acting on  $\mathbb{R}^2$ , the infinite dihedral group  $D_\infty$  acting on  $\mathbb{R}$ , the symmetric group  $S_n$  acting on  $\mathbb{R}^n$  or on  $\mathbb{R}^{n-1}$ , and the triangle groups generated by reflections in the geodesics containing the sides of a triangle (with each angle a rational multiple of  $\pi$ ) in the 2-sphere, the Euclidean plane, or 2 dimensional hyperbolic space.

Let  $\Gamma$  be a reflection group acting on  $X$ , and let  $R$  be the set of all reflections in  $\Gamma$ . Note that  $R$  is closed under conjugation in  $\Gamma$  and that  $\Gamma$  acts on mirrors via  $gM_r = M_{grg^{-1}}$  for all  $g \in \Gamma$  and  $r \in R$ .

**Lemma 3.1.1** *If  $r_1, r_2 \in R$  are distinct reflections, then  $M_{r_1} \neq M_{r_2}$ .*

Proof: Suppose  $M_{r_1} = M_{r_2}$ . Let  $x, y \in X$  be points such that  $x$  and  $y$  are separated by  $M_{r_1}$ . Given any path  $\alpha : [0, 1] \longrightarrow X$  with  $\alpha(0) = x$  and  $\alpha(1) = y$ , the image of  $\alpha$  must meet the mirror  $M_{r_1} = M_{r_1} \cap M_{r_2}$ , contradicting R3.  $\square$

A *chamber* is the closure of a component of the complement

$$X - \bigcup_{r \in R} M_r$$

of all mirrors in  $X$ . Since  $\Gamma$  acts on mirrors, it follows that  $\Gamma$  acts on chambers. Fix a chamber  $Q$  and call it the *fundamental chamber*. Let  $V$  be the set of  $v \in R$  such that

$$\left( M_v - \bigcup_{r \neq v} M_r \right) \cap Q \neq \emptyset.$$

The set  $V$  is called a set of *fundamental reflections* of  $\Gamma$ .

**Lemma 3.1.2** *The subgroup  $\Gamma_V \subseteq \Gamma$  generated by  $V$  acts transitively on chambers.*

Proof: Let  $Q'$  be a chamber. Let  $x \in \text{int}(Q)$  and  $y \in \text{int}(Q')$  (where  $Q$  is the fundamental chamber). Note that for any chamber  $Q'$ ,  $\overline{\text{int}(Q')} = Q'$ . Since  $X$  is path connected, there is a path  $\alpha : [0, 1] \rightarrow X$  with  $\alpha(0) = x$  and  $\alpha(1) = y$ . By R3, we may choose  $\alpha$  so that the image of  $\alpha$  misses the pairwise intersections of mirrors.

Claim 1: If  $r \in \Gamma$  is a reflection, then we may choose the path  $\alpha$  so that the image of  $\alpha$  meets the mirror  $M_r$  in a finite number of points and, at each intersection point,  $\alpha$  crosses from one chamber into another.

Proof: Fix a reflection  $r \in \Gamma$ . Since  $\alpha$  misses pairwise intersections of mirrors, for each point  $q \in \alpha^{-1}(M_r)$ , the point  $\alpha(q)$  meets only the mirror  $M_r$ . Since the collection of mirrors is locally finite in  $X$ , the union  $\bigcup_{s \neq r} M_s$  is closed in  $X$  (the union of a locally finite family of closed subspaces is closed) so, by A3 there exists a neighborhood  $U_{\alpha(q)}$  of  $\alpha(q)$  which misses all mirrors but  $M_r$ , is separated into two components  $U_1$  and  $U_2$ , and the components are interchanged under reflection by  $r$ . This neighborhood determines two chambers  $Q_1$  and  $Q_2$  such that  $U_1 \subseteq Q_1$  and  $U_2 \subseteq Q_2$ . Since  $Q_1 \cup Q_2$  is closed in  $X$ , the inverse image  $\alpha^{-1}(Q_1 \cup Q_2)$  is closed in  $[0, 1]$ . Let

$$a = \inf\{x \in [0, 1] : x \in \alpha^{-1}(Q_1 \cup Q_2)\}$$

and let

$$b = \sup\{x \in [0, 1] : x \in \alpha^{-1}(Q_1 \cup Q_2)\}.$$

Thus for all  $t \in [0, a) \cup (b, 1]$ , the point  $\alpha(t)$  is contained in the complement  $X - (Q_1 \cup Q_2)$ . Since  $Q_1 \cup Q_2$  is path connected, we may choose a path

$\beta : [a, b] \longrightarrow Q_1 \cup Q_2$  joining  $\alpha(a)$  and  $\alpha(b)$ . Note that  $\beta$  may be chosen so that if  $s \in \Gamma$  is a reflection distinct from  $r$ , then the image  $\beta(a, b)$  of the open interval  $(a, b)$  is disjoint from  $M_s$ . If  $\alpha(a)$  and  $\alpha(b)$  are contained in a single chamber (say  $Q_1$ ), then we can replace  $\alpha$  with a path which does not meet  $M_r$  within  $Q_1 \cup Q_2$ . If  $\alpha(a)$  and  $\alpha(b)$  are separated by  $M_r$  then, by A3, we can replace  $\alpha$  with a path which meets  $M_r$  in a single point within  $Q_1 \cup Q_2$ . In either case we can assume that the path  $\alpha$  meets the union  $Q_1 \cup Q_2$  once and, within the union, meets the mirror  $M_r$  in at most one point. At the intersection point,  $\alpha$  crosses from  $Q_1$  into  $Q_2$ . Consequently, for any reflection  $r \in \Gamma$ , the intersection of the image of  $\alpha$  with  $M_r$  is a discrete subset of  $X$ . Since  $M_r$  is closed in  $X$ , the inverse image  $\alpha^{-1}(M_r)$  is a closed, discrete (hence finite) subset of  $[0, 1]$ . In other words, We may choose a path  $\alpha$  which meets each mirror a finite number of times, completing the proof of Claim 1.

Since the image of  $\alpha$  is compact and the collection of mirrors is locally finite in  $X$ , the path  $\alpha$  meets only finitely many mirrors. Thus  $\alpha$  determines a sequence

$$Q = Q_0, Q_1, \dots, Q_n = Q'$$

of chambers that it successively crosses into (repetitions are allowed). The point  $z \in X$  where  $\alpha$  crosses from  $Q_0$  into  $Q_1$  is contained in  $M_v$  for some  $v \in V$ .

Claim 2: If  $M_v$  is the mirror which  $\alpha$  crosses from  $Q_0$  into  $Q_1$  at the point  $z$ , then  $vQ_0 = Q_1$ .

Proof: Since the collection  $\{M_r\}_{r \in R}$  of mirrors in  $X$  is a locally finite family of closed subspaces of  $X$ , the union  $\bigcup_{r \neq v} M_r$  of mirrors distinct from  $M_v$  is closed in  $X$ . Thus there is an open neighborhood  $W$  of  $z$  such that

the intersection

$$W \cap \bigcup_{r \neq v} M_r$$

is empty. By A3 there exists an open neighborhood  $U_z$  of  $z$  such that  $U_z$  is contained in  $W$ , the mirror  $M_v$  separates  $U_z$  into two components  $U_0$  and  $U_1$ , and  $vU_0 = U_1$ . Since  $W$  meets only the wall  $M_r$ , it follows that  $W$  is contained in  $Q_0 \cup Q_1$ . Consequently, we have  $U_0 \subseteq \text{int}Q_0$  and  $U_1 \subseteq \text{int}Q_1$ . Since  $vU_0 \subseteq \text{int}Q_1$ , it follows that

$$v \cdot \text{int}Q_0 \cap \text{int}Q_1 \neq \emptyset.$$

Since  $v$  acts on  $X$  by homeomorphism,  $v$  permutes the components of the complement  $X - \bigcup_{r \in R} M_r$  of mirrors in  $X$ . From this we conclude that  $v \cdot \text{int}Q_0 = \text{int}Q_1$  (if two components of a space have nonempty intersection, then they are equal). Since  $v$  is a homeomorphism of  $X$  we have

$$vQ_0 = v \cdot \overline{\text{int}Q_0} = \overline{v \cdot \text{int}Q_0} = \overline{\text{int}Q_1} = Q_1,$$

completing the proof of the claim.

Now consider  $v_1^{-1}Q_1 = Q_0$  and  $v_1^{-1}Q_2$ . By the preceding argument there is a  $v_2 \in V$  such that  $v_2Q_0 = v_1^{-1}Q_2$ . This implies that  $v_1v_2Q_0 = Q_2$ . Proceeding inductively, we find  $v_1, v_2, \dots, v_n \in V$  such that  $v_1v_2 \cdots v_nQ = Q'$ .  $\square$

Let  $Q'$  be a chamber,  $M_r$  be a mirror, and define

$$Q'_r = Q' \cap \left( M_r - \bigcup_{v \neq r} M_v \right).$$

If  $Q'_r$  is nonempty, then the closure  $\overline{Q'_r}$  of  $Q'_r$  in  $X$  is called a *panel*. Two distinct chambers  $Q_1$  and  $Q_2$  are *adjacent* if they share a panel. A *gallery* of length  $k$  is a sequence of chambers  $Q_0, Q_1, \dots, Q_k$  such that consecutive chambers are adjacent.

**Lemma 3.1.3** *If  $r \in \Gamma$  is a reflection, then there exists an element  $g \in \Gamma_V$  such that  $g^{-1}rg \in V$ .*

Proof: Let  $x, y \in X$  be points in  $X$  which are separated by  $M_r$ . Consider a path  $\alpha : [0, 1] \rightarrow X$  in  $X$  with  $\alpha(0) = x$  and  $\alpha(1) = y$ , missing pairwise intersections of mirrors. If  $z$  is a point in which the image of  $\alpha$  meets  $M_r$ , then by A3 there is an open neighborhood  $U_z$  which is separated into two components  $U_1$  and  $U_2$  by  $M_r$  with  $U_1$  and  $U_2$  interchanged by the reflection  $r$ . As in the proof of Claim 2 in Lemma 3.1.2 we may assume that  $U_z$  is chosen so that  $U_1$  is contained in a single chamber  $Q'$ . From this we conclude that the chamber  $Q'$  meets the mirror  $M_r$  in a panel. By Lemma 3.1.2,  $Q' = gQ$  for some  $g \in \Gamma_V$ . Thus  $g^{-1}rg$  is a reflection whose mirror  $g^{-1}M_r = M_{g^{-1}rg}$  contains a panel in  $Q$ , implying that  $g^{-1}rg \in V$ .  $\square$

**Theorem 3.1.4** *If  $V$  is a set of fundamental reflections for  $\Gamma$ , then  $\Gamma_V = \Gamma$ .*

Proof: By R1, the group  $\Gamma$  is generated by reflections. By Lemma 3.1.3, every reflection is an element of  $\Gamma_V$ .  $\square$

**Lemma 3.1.5** *If  $v, w \in V$  and the chambers  $gvQ$  and  $gQ$  are on opposite sides of  $M_w$ , then  $gv = wg$ .*

Proof: Applying  $g^{-1}$  we see that  $Q$  and  $vQ$  are on opposite sides of the mirror  $g^{-1}M_w = M_{g^{-1}wg}$ . Since  $M_v$  meets the (adjacent) chambers  $Q$  and  $vQ$  in a panel, it follows that  $M_v = M_{g^{-1}wg}$ . By Lemma 3.1.1,  $v = g^{-1}wg$ .  $\square$

The *length* of  $g \in \Gamma$  is

$$l_V(g) = \min\{n : g = v_1v_2 \cdots v_n, v_i \in V\}.$$

We define the length of the identity element  $1 \in \Gamma$  to be  $l_V(1) = 0$ . If  $g = v_1v_2 \cdots v_n$  where  $v_i \in V$  for  $1 \leq i \leq n$  and  $l_v(g) = n$ , then the word  $v_1v_2 \cdots v_n$  is

called a *reduced expression* or a *reduced word* for  $g$ . Note that  $l_V(g)$  is the length of the shortest gallery

$$Q, v_1Q, v_1v_2Q, \dots, v_1v_2 \cdots v_nQ = gQ$$

joining  $Q$  and  $gQ$ .

**Lemma 3.1.6** *If  $g \in \Gamma$ , then  $l_V(g) = 1$  if and only if  $g \in V$ .*

Proof: If  $l_V(g) = 1$ , then there is some  $v \in V$  such that  $g = v$ , implying that  $g \in V$ .

To prove the reverse implication, we suppose that  $g \in V$  (in which case we have  $l_V(g) \leq 1$ ). Since  $g$  interchanges the components of the complement  $X - M_g$ , the action of  $g$  on  $X$  is nontrivial. This implies that  $g \neq 1$  and therefore  $l_V(g) = 1$ . □

**Lemma 3.1.7** *The reflection group  $\Gamma$  acts freely on chambers (i.e., for any chamber  $Q'$ , if  $gQ' = Q'$ , then  $g = 1$ ).*

Proof: Let  $Q'$  be any chamber and let  $g \in \Gamma - \{1\}$  be a nontrivial element of  $\Gamma$  which leaves  $Q'$  invariant. By Lemma 3.1.6, we have  $l_V(g) = n$  for some integer  $n > 1$ . Thus we may express  $g$  as a word  $g = v_1v_2 \cdots v_n$ , where  $v_i \in V$  for  $1 \leq i \leq n$ . Consider the gallery

$$Q, v_1Q, v_1v_2Q, \dots, v_1v_2 \cdots v_nQ = Q,$$

and let  $i \geq 1$  be such that  $v_1 \cdots v_iQ$  and  $v_1 \cdots v_i v_{i+1}Q$  are separated by  $M_{v_1}$ .

By Lemma 3.1.5,

$$(v_1 \cdots v_i)v_{i+1} = v_1(v_1 \cdots v_i) = v_2 \cdots v_i,$$

contradicting the assumption that  $l_V(g) = n$ . □

The following lemma is a useful characterization of the length of the elements of  $\Gamma$  in terms of galleries in  $X$ .

**Lemma 3.1.8** *If  $g = v_1v_2 \cdots v_n$ , then  $l_V(g) = n$  if and only if the gallery*

$$Q, v_1Q, v_1v_2Q, \dots, v_1v_2 \cdots v_nQ$$

*never crosses the same mirror twice.*

Proof: Assume that  $l_V(g) = n$  and suppose that for some positive integers  $i$  and  $j$  with  $1 \leq i < j \leq n$ , the chambers

$$v_1v_2 \cdots v_iv_{i+1}Q \text{ and } v_1v_2 \cdots v_i \cdots v_jQ$$

are on the same side of the mirror  $v_1v_2 \cdots v_iM_{v_{i+1}}$ , while the chambers

$$v_1v_2 \cdots v_iv_{i+1}Q \text{ and } v_1v_2 \cdots v_i \cdots v_jv_{j+1}Q$$

are separated by the mirror  $v_1v_2 \cdots v_iM_{v_{i+1}}$  (in other words, the gallery

$$Q, v_1Q, \dots, v_1v_2 \cdots v_nQ$$

crosses over the mirror  $v_1 \cdots v_iM_{v_{i+1}}$  at least twice). Applying  $(v_1v_2 \cdots v_i)^{-1}$  we see that the chambers

$$v_{i+1} \cdots v_jQ \text{ and } v_{i+1} \cdots v_jv_{j+1}Q$$

are separated by the mirror  $M_{v_{i+1}}$ . By Lemma 3.1.5, it follows that

$$v_{i+1}(v_{i+1} \cdots v_j) = (v_{i+1} \cdots v_j)v_{j+1}$$

and so

$$g = v_1 v_2 \cdots v_i v_{i+2} \cdots v_j v_{j+2} \cdots v_n,$$

contradicting the assumption that  $l_V(g) = n$ .

For the reverse implication we assume that the gallery

$$Q, v_1 Q, v_1 v_2 Q, \dots, v_1 v_2 \cdots v_n Q$$

never crosses the same mirror twice and observe that if  $g = u_1 \cdots u_m$  is another word representing  $g$ , then the gallery

$$Q, u_1 Q, \dots, u_1 u_2 \cdots u_m Q$$

must cross each of the walls which the gallery

$$Q, v_1 Q, v_1 v_2 Q, \dots, v_1 v_2 \cdots v_n Q$$

crosses. This implies that  $m \geq n = l_V(g)$  and so there is no expression for  $g$  consisting of fewer than  $n$  generators.  $\square$

In light of Lemma 3.1.8, we may view  $l_V(g)$  as the number of mirrors separating  $Q$  from  $gQ$  in a gallery of shortest length.

The essence of the theory of reflection groups is contained in the following theorem.

**Theorem 3.1.9** (The Exchange Principle) *Let  $g \in \Gamma$  with  $l_V(g) = n$ , and let*

$$g = v_1 \cdots v_n = u_1 \cdots u_n$$

*be two shortest word representatives for  $g$ . If  $u_1 \neq v_1$ , then there is an  $i \geq 1$  such that*

$$g = u_1 v_1 \cdots v_{i-1} v_{i+1} \cdots v_n.$$

Proof: By Lemma 3.1.8, there is an  $1 \leq i \leq n$  such that  $v_1 \cdots v_{i-1}Q$  and  $v_1 \cdots v_iQ$  are on opposite sides of  $M_{u_1}$ . By Lemma 3.1.5,

$$u_1v_1 \cdots v_{i-1} = v_1 \cdots v_i,$$

and the result follows.  $\square$

If  $u, v \in V$ , let  $m(u, v)$  be the order of  $uv$  in  $\Gamma$ . Note that  $m(u, v) \in \mathbb{N} \cup \{\infty\}$  for any  $u, v \in V$ . The proof of the following may be found in [22], (Theorem 1.9), and is accomplished by repeated applications of the Exchange Principle.

**Theorem 3.1.10** *A presentation for  $\Gamma$  is given by*

$$\langle V : (uv)^{m(u,v)}, u, v \in V \rangle .$$

Note that the relation  $v^2 = 1$  implies that  $m(u, v) = m(v, u)$ . Relations of the form  $(uv)^\infty$  are discarded.

### 3.2 Coxeter Groups

For the reader's convenience, we restate the following definition. A *Coxeter system* is a triple  $(W, S, m)$  where  $W$  is a group,  $S \subseteq W$ , and

$$m : S \times S \longrightarrow \mathbb{N} \cup \{\infty\}$$

such that

1. for all  $s, t \in S$ ,  $m(s, t) = 1$  if and only if  $s = t$ ;
2.  $m(s, t) = m(t, s)$  for all  $s, t \in S$ ;

3.  $W$  has a presentation of the form

$$\langle S : (st)^{m(s,t)} ; s, t \in S \rangle .$$

$W$  is called a *Coxeter group*. When there is no confusion, the “ $m$ ” will often be omitted from the notation and the pair  $(W, S)$  will be referred to as a Coxeter system. As in the case of reflection groups, relations of the form  $(st)^\infty$  are discarded. We assume throughout that  $W$  is a finitely generated Coxeter group (i.e.,  $S$  is a finite set).

By Theorem 3.1.10, every reflection group is a Coxeter group.

Let

$$R_S = \{wsw^{-1} : w \in W, s \in S\}.$$

If  $r \in R_S$ , the element  $r$  is called a *reflection*.

Given a Coxeter system  $(W, S)$ , the *Coxeter graph*  $\Gamma_S$  is the labeled graph defined as follows:

1. the vertex set of  $\Gamma_S$  is  $S$ ;
2. vertices  $s, t \in S$  are connected by an edge  $\{s, t\}$  if and only if  $m(s, t) \geq 3$ ;
3. the edge  $\{s, t\}$  is labeled with  $m(s, t)$  if and only if  $m(s, t) \geq 4$ .

The Coxeter graph  $\Gamma_S$  provides a convenient way of representing the Coxeter system  $(W, S)$ . Coxeter graphs  $\Gamma_S$  and  $\Gamma_{S'}$ , are *graph-isomorphic* if there is a bijection  $\theta : S \rightarrow S'$  such that

1.  $\{s, t\}$  is an edge in  $\Gamma_S$  if and only if  $\{\theta(s), \theta(t)\}$  is an edge in  $\Gamma_{S'}$ ;
2.  $m(s, t) = m'(\theta(s), \theta(t))$ .

We now collect some standard results and terminology from the theory of Coxeter groups. Given a Coxeter system  $(W, S)$  and  $w \in W$ , the *length* of  $w$  is

$$l_S(w) = \min\{n : w = s_1 s_2 \cdots s_n, s_i \in S\}.$$

If  $w = s_1 s_2 \cdots s_n$  and  $l(w) = n$ , then  $s_1 s_2 \cdots s_n$  is called a *reduced expression* for  $w$ . In other words,  $l_S(w)$  is the minimal number of generators required in a reduced expression for  $w$ . When there is no confusion, the “ $S$ ” will be deleted from the notation, and we write  $l(w)$  for the length of  $w$ .

**Lemma 3.2.1** *Let  $u, v \in W$  and  $s \in S$ .*

- a)  $l(u) = l(u^{-1})$ .
- b)  $l(u) = 1$  if and only if  $u \in S$ .
- c)  $l(uv) \leq l(u) + l(v)$ .
- d)  $l(uv) \geq l(u) - l(v)$ .
- e)  $l(u) - 1 \leq l(su) \leq l(u) + 1$ .

Proof: Assume that each  $s_i \in S$ .

- a) If  $u = s_1 s_2 \cdots s_k$ , then  $u^{-1} = s_k \cdots s_2 s_1$ . This implies that  $l(u^{-1}) \leq l(u)$  for all  $u \in W$ . In particular,

$$l(u) = l((u^{-1})^{-1}) \leq l(u^{-1}) \leq l(u).$$

This implies that  $l(u) = l(u^{-1})$ .

- b) This follows directly from Theorem 3.3.2, stated below, and Lemma 3.1.6.

c) Suppose  $u = s_1 \cdots s_i$  and  $v = s'_1 \cdots s'_j$  are reduced expressions for  $u$  and  $v$ . Then  $uv = s_1 \cdots s_i s'_1 \cdots s'_j$  has length less than equal to  $i + j$ .

d) By a and c,

$$l(u) = l(uvv^{-1}) \leq l(uv) + l(v^{-1}) = l(uv) + l(v).$$

e) This follows directly from b, c and d.

If  $T \subseteq S$ , write  $W_T$  for the subgroup of  $W$  generated by  $T$ . Define  $W_\emptyset$  to be the trivial group (i.e.,  $W_\emptyset = \{1\}$ ). If  $W_T$  is finite,  $W_T$  is called a *special subgroup*. If  $W_T$  is a special subgroup and  $w \in W$ , the conjugate subgroup  $wW_Tw^{-1}$  is called a *parabolic subgroup*. If  $G$  is a parabolic subgroup of  $W$ , define the *parabolic rank*,  $rk(G)$ , to be  $|T|$ , where  $T \in \mathcal{S}(S)$  is such that  $G = wW_Tw^{-1}$  for some  $w \in W$ . Let  $P(S)$  be the set of all parabolic subgroups of  $W$  and let  $P_k(S)$  be the set of parabolic subgroups of  $W$  with parabolic rank equal to  $k$ .

**Theorem 3.2.2** (see [22], Theorem 5.5 (a)) *If  $T \subseteq S$  then  $W_T$  has a presentation of the form*

$$\langle T : (st)^{m(s,t)}, s, t \in T \rangle .$$

Thus, for any  $T \subseteq S$ ,  $W_T$  is also a Coxeter group. With this in mind, for any  $w \in W_T$  we may define

$$l_T(w) = \min\{n : w = t_1 t_2 \cdots t_n, t_i \in T\}.$$

Some important facts about special subgroups of Coxeter groups are contained in the following theorems.

**Theorem 3.2.3** (see [22], Theorem 5.5 (b)) *Let  $T \subseteq S$ . If  $w \in W_T$ , then  $l_T(w) = l_S(w)$ .*

A consequence of Theorem 3.2.3 is the following corollary.

**Corollary 3.2.4** *Let  $A, B \subseteq S$ . If  $W_A = W_B$ , then  $A = B$ .*

Proof: Let  $a \in A$ . By Lemma 3.2.1 b,  $l_A(a) = 1$ , so we have

$$1 = l_A(a) = l_S(a) = l_B(a),$$

by Theorem 3.2.3. Applying 3.2.1 b, we see that  $a \in B$  and so  $A \subseteq B$ . The same argument shows that  $B \subseteq A$  and hence,  $A = B$ .  $\square$

The following result is particularly useful in the theory of Coxeter groups.

**Theorem 3.2.5** (see [22], Corollary 5.10 (c)) *If  $A, B \subseteq S$ , then*

$$W_A \cap W_B = W_{A \cap B}.$$

### 3.3 The Geometric Representation

We now discuss some useful ways of representing Coxeter groups as groups that act on vector spaces or subsets of vector spaces. In particular, we will see that Coxeter groups are reflection groups.

Fix a Coxeter system  $(W, S, m)$ , where  $n = |S|$ , and let  $V$  be a vector space over the field  $\mathbb{R}$  with basis  $\{e_s\}_{s \in S}$  in one-to-one correspondence with the elements of  $S$ . Let

$$B : V \times V \longrightarrow \mathbb{R}$$

be the bilinear form on  $V$  defined by

$$B(e_s, e_t) = -\cos\left(\frac{\pi}{m(s, t)}\right)$$

for all  $s, t \in S$ . If  $m(s, t) = \infty$ , then define  $B(e_s, e_t) = -1$ . The bilinear form  $B$  is called the *canonical bilinear form* associated with the Coxeter system  $(W, S)$ .

Let

$$\alpha : W \longrightarrow GL_n(V)$$

be the map defined by

$$\alpha(s)(x) = x - 2B(e_s, x)e_s$$

for all  $s \in S$  and all  $x \in V$ . Thus, for each  $s \in S$ , the image  $\alpha(s)$  acts by reflection on  $V$  fixing the hyperplane  $H_s$  orthogonal to  $e_s$ .

The following results are due to J. Tits.

**Theorem 3.3.1** (J. Tits, [6], p. 93) *The map  $\alpha$  is a faithful representation of the Coxeter group  $W$  into  $GL_n(V)$  (where  $GL_n(V)$  is the group of nonsingular linear transformations of  $V$ ).*

The homomorphism  $\alpha$  is called the *geometric representation* of  $W$ . This allows us to view each  $r \in R_S$  as a reflection on the vector space  $V$  fixing a hyperplane  $H_r$  in  $V$ . Unfortunately, the collection  $\{H_r\}_{r \in R_S}$  may not be locally finite, so  $W$  may not act as a reflection group on  $W$ . The following theorem remedies the situation.

**Theorem 3.3.2** (J. Tits) *Let  $V^*$  be the dual space to  $V$ . There exists a faithful representation*

$$\alpha^* : W \longrightarrow GL_n(V^*)$$

*and an open convex set  $C \subseteq V^*$ , invariant under  $\alpha^*(W)$ , such that  $\alpha^*(W)$  acts as a reflection group on  $C$ .*

In light of Theorem 3.3.2, we may regard any Coxeter group as a reflection group. The subset  $C$  of  $V^*$  is called the *Tits cone*. For discussion and proofs of Theorems 3.3.1 and 3.3.2, see [22], Corollary 5.4 and Theorem 5.13, respectively.

A Coxeter system  $(W, S)$  is said to be *irreducible* if the Coxeter graph  $\Gamma_S$  is connected. It is appropriate to note here that all irreducible systems  $(W, S)$  with finite  $W$  have been classified. In other words, there is an (infinite) list of Coxeter graphs such that, if  $W$  is a Coxeter group with irreducible Coxeter system  $(W, S)$ , then the Coxeter graph  $\Gamma_S$  appears in the list (cf. [22], Chapter 2). The canonical bilinear form  $B$  is instrumental in this classification.

Given a real vector space  $W$  and bilinear form  $A : W \times W \rightarrow \mathbb{R}$ , the form  $A$  is said to be *positive definite* if  $A(w, w) > 0$  for all nonzero  $w \in W$ . The following theorem is due to Witt [32].

**Theorem 3.3.3** (Witt [32]) *Let  $(W, S)$  be a Coxeter system. Then  $W$  is finite if and only if the canonical bilinear form  $B$  associated with  $(W, S)$  is positive definite.*

A connected graph  $T$  is a *tree* if  $T$  contains no cycles. With this definition in hand, we use Theorem 3.3.3 to prove:

**Corollary 3.3.4** *If  $(W, S)$  is an irreducible Coxeter system with  $W$  finite, then the Coxeter graph  $\Gamma_S$  is a tree.*

Proof: Assume  $W$  is finite with irreducible Coxeter system  $(W, S)$ . Enumerate  $S$  as  $S = \{s_1, \dots, s_k\}$ . Suppose  $v_1, v_2, \dots, v_n$  are the vertices of a cycle in the Coxeter graph  $\Gamma_S$ . Then  $m(v_i, v_{i+1}) \geq 3$  for  $1 \leq i \leq n$  (where  $v_{n+1} \equiv v_1$ ). Let  $V$  be a real vector space with basis  $\{e_{s_1}, e_{s_2}, \dots, e_{s_k}\}$  in one-to-one correspondence with  $S$  and we assume without loss of generality that  $e_{s_i} = e_{v_i}$  for  $1 \leq i \leq n$ .

Let  $x \in V$  be the vector

$$x = e_{s_1} + e_{s_2} + \cdots + e_{s_n}.$$

By direct computation we see that

$$B(x, x) = n + 2 \sum_{i < j} B(e_{s_i}, e_{s_j}).$$

Since  $B(e_{s_i}, e_{s_{i+1}}) \leq -\cos \frac{\pi}{3} = -\frac{1}{2}$  for  $1 \leq i \leq n$  and  $B(e_{s_i}, e_{s_j}) \leq 0$  for  $i < j$ , it follows that  $B(x, x) \leq 0$ . By Theorem 3.3.3,  $W$  is infinite, contradicting our assumption that  $W$  is finite.  $\square$

Corollary 3.3.4 allows one to detect some infinite Coxeter groups by direct inspection of the Coxeter graph: if the graph contains a cycle, it is necessarily infinite.

### 3.4 Simplicial Complexes Associated with Coxeter Groups

We now describe the constructs which provide the link between the theory of Coxeter groups and geometry. Much of the material in this section follows R. Charney and M. Davis, [11], [12], and [13]. Details on abstract simplicial complexes may be found in [25].

Given a set  $X$ , a family  $\mathcal{A}$  of finite subsets of  $X$  is an *abstract simplicial complex* if, whenever  $A \in \mathcal{A}$  and  $B \subseteq A$ , then  $B \in \mathcal{A}$ . If  $A \in \mathcal{A}$ , and  $|A| = k + 1$ ,  $A$  is called a *k-simplex*. A singleton in  $\mathcal{A}$  is called a *vertex*. Denote the collection of vertices of  $\mathcal{A}$  by  $V(\mathcal{A})$ . An abstract simplicial complex  $\mathcal{A}$  may be represented as a simplicial complex (called the *geometric realization* and denoted  $geom(\mathcal{A})$ ) in the topological sense in the following manner: the vertex set of  $geom(\mathcal{A})$  is in one-to-one correspondence with the set of singletons in  $\mathcal{A}$ . The set of edges is in one-to-one correspondence with the two-element sets in  $\mathcal{A}$ . Proceeding

inductively, the set of  $k$ -dimensional simplices is in one-to-one correspondence with the elements of order  $k + 1$  in  $\mathcal{A}$ . The space  $geom(\mathcal{A})$  is given the “weak topology” with respect to its simplices.

If  $\mathcal{A}$  is an abstract simplicial complex with  $A \in \mathcal{A}$ , and  $B \subseteq A$  with  $|B| = k + 1$ , the simplex  $B$  is called a  $k$ -dimensional face of  $A$ . If  $A, B \in \mathcal{A}$  have dimension  $n$ , the simplices  $A$  and  $B$  are said to be *adjacent* if they share an  $(n - 1)$ -dimensional face. A simplex  $A$  is *maximal* in  $\mathcal{A}$  if it is not properly contained in any other simplex of  $\mathcal{A}$ . Abstract simplicial complexes  $\mathcal{A}$  and  $\mathcal{B}$  are *isomorphic* if there is a bijection

$$\Psi : V(\mathcal{A}) \longrightarrow V(\mathcal{B})$$

such that

$$A = \{a_0, a_1, \dots, a_k\}$$

is a  $k$ -simplex in  $\mathcal{A}$  if and only if

$$\{\Psi(a_0), \Psi(a_1), \dots, \Psi(a_k)\}$$

is a  $k$ -simplex in  $\mathcal{B}$ .

If  $\mathcal{P}$  is a partially ordered set (poset), let  $\mathcal{P}'$  be the set of all finite chains of  $\mathcal{P}$ . The poset  $\mathcal{P}'$ , partially ordered by inclusion (of finite chains) is an abstract simplicial complex ( $\mathcal{P}'$  is called the *derived poset* of  $\mathcal{P}$ ). Thus there is a  $k$ -simplex in  $geom(\mathcal{P}')$  for every chain

$$x_0 < x_1 < \dots < x_k$$

of length  $k + 1$  in  $\mathcal{P}$ . If  $\alpha \in \mathcal{P}$ , let  $\mathcal{P}_{\alpha \geq}$  be the subposet of all elements in  $\mathcal{P}$  less than or equal to  $\alpha$ . Define the subposets  $\mathcal{P}_{\alpha >}$ ,  $\mathcal{P}_{\alpha \leq}$ , and  $\mathcal{P}_{\alpha <}$  similarly.

For the remainder of this section, let  $(W, S)$  be a fixed Coxeter system. Let

$$\mathcal{S}(S) = \{T \subseteq S : W_T \text{ is finite}\}$$

be the set of subsets of  $S$  which generate special subgroups of  $W$ . With a partial ordering of set-theoretic inclusion,  $\mathcal{S}(S) - \emptyset$  is an abstract simplicial complex. Let  $N(S)$  denote its geometric realization.  $N(S)$  is called the *nerve* corresponding to the system  $(W, S)$ . Thus there is a vertex in  $N(S)$  for each element  $s \in S$ , and edge in  $N(S)$  for each pair  $\{s, t\} \in \mathcal{S}(S)$ , etc. Let  $N^*(S)$  denote the collection of maximal simplices of  $N(S)$  (i.e.,  $N^*(S)$  is the set of all simplices of the nerve  $N(S)$  which are not properly contained in another simplex of  $N(S)$ ).

M. Davis has constructed a simplicial complex  $\Sigma(S)$  such that  $W$  acts by reflections on  $\Sigma(S)$ . It is described as follows: Let

$$W\mathcal{S}(S) = \{wW_T : w \in W, W_T \in \mathcal{S}(S)\}$$

be the set of all cosets in  $W$  of all special subgroups of  $W$ . Partially order  $W\mathcal{S}(S)$  by set-theoretic inclusion. The following result describes this partial order explicitly.

**Lemma 3.4.1** *If  $uW_A, vW_B \in W\mathcal{S}(S)$ , then  $uW_A \leq vW_B$  if and only if  $A \subseteq B$  and  $v^{-1}u \in W_B$  (where the symbol “ $\leq$ ” denotes the partial ordering on  $W\mathcal{S}(S)$ ).*

Proof: Suppose  $uW_A \leq vW_B$ . This means that  $uW_A \subseteq vW_B$  and therefore  $v^{-1}uW_A \subseteq W_B$ . Since  $v^{-1}u \in v^{-1}uW_A$ , it follows that  $v^{-1}u \in W_B$ . Now assume  $a \in A$ . In this case,  $v^{-1}ua \in W_B$  and, since  $(v^{-1}u)^{-1} \in W_B$ , we have

$$a = (v^{-1}u)^{-1} \in (v^{-1}u)^{-1}W_B = W_B.$$

Therefore  $a \in W_B$ . By Lemma 3.2.3,

$$1 = l_A(a) = l_S(a) = l_B(a).$$

By Lemma 3.2.1 b, this is possible if and only if  $a \in B$ . This implies that  $A \subseteq B$ .

For the reverse implication, suppose  $A \subseteq B$  and  $v^{-1}u \in W_B$ . Then

$$W_A \subseteq W_B = u^{-1}vW_B.$$

It follows that  $uW_A \subseteq vW_B$ . □

Define

$$\Sigma(S) = \text{geom}((W\mathcal{S}(S))')$$

to be the geometric realization of the derived poset of  $W\mathcal{S}(S)$ . Thus there is a  $k$ -simplex in  $\Sigma(S)$  for each chain of the form

$$w_0W_{T_0} < w_1W_{T_1} < \cdots < w_kW_{T_k}.$$

**Theorem 3.4.2** (M. Davis, [13])  *$W$  acts simplicially on  $\Sigma$  by left translation. Moreover, if  $r \in R_S$  then  $r$  acts by reflection on  $\Sigma$ .*

The complex  $\Sigma(S)$  is called the *Davis complex* corresponding to  $(W, S)$ . When there is no confusion we delete the “ $S$ ” from the notation and write  $\Sigma$  for the Davis complex. Let

$$K = \text{geom}((W\mathcal{S}(S))'_{W_{\emptyset \leq}}).$$

In other words,  $K$  is the geometric realization of the derived subposet of elements in  $W\mathcal{S}(S)$  which are greater than or equal to  $W_{\emptyset}$ . The subcomplex  $K$  is the *fundamental chamber* of  $\Sigma$  (note that  $K$  is isomorphic to  $\text{geom}((\mathcal{S}(S))')$ ).

Theorem 3.4.2 provides another example of the fact that all Coxeter groups are reflection groups.

The link between the theory of Coxeter groups and geometry is provided by the following result.

**Theorem 3.4.3** (G. Moussong, [24]) *There is a piecewise Euclidean metric  $d_M$  on  $\Sigma$  such that  $(\Sigma, d_M)$  is a CAT(0) space. Moreover, the action of  $W$  on  $\Sigma$  is proper, cocompact and by isometries (hence Coxeter groups are CAT(0) groups).*

The metric  $d_M$  is called the *Moussong metric*. Combining Theorem 3.4.3 with Theorem 2.4.5, we have:

**Corollary 3.4.4** *Coxeter groups have solvable conjugacy problem.*

Prior to Moussong’s result, K. Appel and P. Schupp [3] had shown that Coxeter groups of “extra-large type” (i.e.,  $m(s, t) \geq 4$  when  $s \neq t$ ) have solvable conjugacy problem. Moussong’s result provides the only known solution to the general question of conjugacy in Coxeter groups.

## 4 The Rigidity Question

### 4.1 The Davis Complex Revisited

In this section we fix an arbitrary Coxeter system  $(W, S)$ . Recall that if  $r \in R_S$  is a reflection in  $W$ , then  $r$  acts by isometry on the Davis complex  $\Sigma(S)$  as a reflection. Denote the wall corresponding to  $r$  by  $\Sigma(S)^r$ . More generally, if  $H \leq W$  is a finite subgroup of  $W$ , we define

$$\Sigma(S)^H = \{x \in \Sigma(S) : gx = x \text{ for all } g \in H\}$$

to be the subset of  $\Sigma(S)$  which is fixed point-wise by each element of  $H$  (recall that by Theorem 2.4.3,  $H$  must fix some point of  $\Sigma(S)$ ). Once again, when there is no confusion we delete the “ $S$ ” from the notation and write  $\Sigma^H$  for the set of points in  $\Sigma$  fixed by  $H$ .

The following result characterizes the nature of the action of infinite subgroups of  $W$  on  $\Sigma$ .

**Lemma 4.1.1** *If  $H$  is an infinite subgroup of  $W$ , then  $\Sigma^H = \emptyset$ .*

Proof: We prove the contrapositive. Suppose  $\Sigma^H \neq \emptyset$  and let  $x \in \Sigma^H$  be a point of  $\Sigma$  fixed by  $H$ . By Theorem 3.4.3, the action of  $W$  on  $\Sigma$  is proper which implies that  $stab_W(x)$  is finite. Since  $H$  is a subgroup of  $stab_W(x)$  it follows that  $H$  is finite. □

The following result may be found in [11].

**Lemma 4.1.2** (R. Charney, M. Davis, [11]) *Let  $G_1, G_2$  be parabolic subgroups of  $W$ .*

- a) *If  $G_1$  and  $G_2$  are finite, then  $G_1 \cap G_2$  is a parabolic subgroup of  $W$ .*
- b) *If  $G_1 \subseteq G_2$  and  $rk(G_1) = rk(G_2)$ , then  $G_1 = G_2$ .*

The next result follows immediately from Theorem 2.4.3 and Lemma 4.1.1.

**Corollary 4.1.3** *Let  $G_1$  and  $G_2$  be parabolic subgroups of  $W$ , and let  $G$  be the subgroup generated by  $G_1 \cup G_2$ . Then  $\Sigma^G = \emptyset$  if and only if  $G$  is infinite.*

## 4.2 Definitions and Results

Recall that a Coxeter group is *rigid* if, for any Coxeter systems  $(W, S)$  and  $(W, S')$  there exists an automorphism  $\rho : W \rightarrow W$  such that  $\rho(S) = S'$ . Note that  $W$  is rigid if and only if  $\Gamma_S$  and  $\Gamma_{S'}$  are graph isomorphic. Since the dihedral group  $D_6$  has presentations

$$\langle x, y : x^2, y^2, (xy)^6 \rangle,$$

as well as

$$\langle a, b, c : a^2, b^2, c^2, (ab)^2, (ac)^2, (bc)^3 \rangle,$$

it is immediate that not all Coxeter groups are rigid (an example of infinite Coxeter group which is not rigid is provided by considering the free product  $\mathbb{Z}_2 * D_6$ ). There are certain classes of Coxeter groups which have been shown to be rigid.

**Theorem 4.2.1** (D. Radcliffe [29]) *If  $(W, S, m)$  is a Coxeter system such that*

$$m(s, t) \in \{2, \infty\} \text{ for all distinct } s, t \in S,$$

*then  $W$  is rigid (a group equipped with such a system is said to be “right-angled”).*

Recall that a simplicial complex  $K$  is an  $n$ -dimensional homology manifold if for every vertex  $v \in K$ ,

$$H_i(K, K - \{v\}) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases},$$

and that  $K$  is a homology  $n$ -sphere if

$$H_i(K) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & \text{if } i \neq n \end{cases}.$$

**Theorem 4.2.2** (R. Charney, M. Davis [11]) *Let  $(W, S)$  be a Coxeter system such that  $N(S)$  is an  $(n - 1)$ -dimensional homology manifold and a homology  $(n - 1)$ -sphere. If  $(W, S')$  is any other Coxeter system for  $W$ , then there exists a unique  $w \in W$  such that  $wSw^{-1} = S'$*

Charney and Davis say that a Coxeter system which satisfies the hypotheses of Theorem 4.2.2 is of “type  $HM_n$ .” In the same paper they prove an analogous result in the case that the nerve  $N(S)$  is an  $(n - 1)$ -dimensional pseudomanifold and  $H_{n-1}(N(S)) \cong \mathbb{Z}$ .

In both the right-angled case and the type  $HM_n$  case we see that the structure of Coxeter graph must be sufficiently restricted in order to obtain rigidity. For example, in the right-angled case, every edge of the graph has label  $\infty$ .

In the setting of Theorem 4.2.2, the homological structure imposed on the nerve induces strict requirements on the isomorphism-type of the Coxeter graph

as well as the labeling scheme. We illustrate this in the case that  $(W, S, m)$  is a type  $HM_2$  Coxeter system. In order to do so we introduce the *modified Coxeter graph*  $\tilde{\Gamma}_S$  defined as follows:

- a) The vertex set of  $\tilde{\Gamma}_S$  is equal to  $S$ .
- b) Each pair of vertices  $s, t \in \tilde{\Gamma}_S$  is joined by an edge labeled by  $m(s, t)$ .

Thus the standard Coxeter graph  $\Gamma_S$  is obtained from the modified Coxeter graph  $\tilde{\Gamma}_S$  by deleting the edges labeled “2” in  $\tilde{\Gamma}_S$  and removing the label from any edge in  $\tilde{\Gamma}_S$  labeled “3.” If  $(W, S)$  is of type  $HM_2$ , the nerve  $N(S)$  is a topological 1-sphere, and the modified Coxeter graph  $\tilde{\Gamma}_S$  has the following form:

1.  $\tilde{\Gamma}_S$  is the complete graph on  $|S|$  vertices (a graph is said to be *complete* if every pair of vertices is joined by an edge);
2. For any vertex  $p$  in  $\tilde{\Gamma}_S$ , the cardinality of the set

$$\{t \in S : m(t, p) < \infty\}$$

is equal to 2.

Generally speaking, the local topology of the Davis complex is complicated. The homological hypotheses of Theorem 4.2.2 have the effect of reducing the complexity of the Davis complex significantly, as the following result illustrates.

**Theorem 4.2.3** (R. Charney, M. Davis, [11]) *Suppose  $(W, S)$  is type  $HM_n$ . Then, for each  $T \in \mathcal{S}(S)$  with  $|T| = k$ ,  $\Sigma^{W_T}$  is a homology  $(n - k)$ -manifold. In particular (for  $T = \emptyset$ ),  $\Sigma$  is a homology  $n$ -manifold.*

Theorem 4.2.3 leads Charney and Davis to conclude:

**Theorem 4.2.4** (R. Charney, M. Davis, [11]) *If  $(W, S)$  and  $(W, S')$  are Coxeter systems for  $W$ , then  $(W, S)$  is type  $HM_n$  if and only if  $(W, S')$  is type  $HM_n$ .*

Thus it makes sense to define a Coxeter group  $W$  to be of *type  $HM_n$*  if any Coxeter system  $(W, S)$  is of type  $HM_n$ .

Recall that for each positive integer  $k$ ,

$$P_k(S) = \{wW_Tw^{-1} \subseteq W : w \in W \text{ and } T \in \mathcal{S}(S) \text{ with } |T| = k\}$$

is the set of  $S$ -parabolic subgroups of  $W$  of rank  $k$ . Theorem 4.2.3 enables Charney and Davis to show the following crucial result (also found in [11]).

**Theorem 4.2.5** (R. Charney and M. Davis, [11]) *If  $W$  is of type  $HM_n$  and  $(W, S)$  and  $(W, S')$  are Coxeter systems for  $W$ , then for each positive integer  $k$ ,  $P_k(S) = P_k(S')$ . In particular, if  $r \in R_{S'}$ , then  $r$  acts by reflection on  $\Sigma(S)$ .*

When pursuing the rigidity question for Coxeter groups, Coxeter systems  $(W, S, m)$  and  $(W, S', m')$  are given and one tries to produce an automorphism  $\rho$  of  $W$  that carries  $S$  to  $S'$ . In particular it is necessary that, for each pair  $s, t \in S$ ,

$$m(s, t) = m'(\rho(s), \rho(t)).$$

We shall see that this consideration partially manifests itself topologically in the Davis complex. By Corollary 4.1.3,  $m(s, t) < \infty$  if and only if

$$\Sigma(S)^s \cap \Sigma(S)^t = \Sigma(S)^{W_{\{s,t\}}} \neq \emptyset.$$

If one knows that  $R_S = R_{S'}$  (as in the  $HM_n$  case), we see that

$$\Sigma(S)^{\rho(s)} \cap \Sigma(S)^{\rho(t)} = \Sigma(S)^{W_{\{\rho(s), \rho(t)\}}} \neq \emptyset$$

is necessary to ensure that  $m'(\rho(s), \rho(t)) < \infty$ . In other words, when it is known that  $\rho(s)$  and  $\rho(t)$  act by reflection on  $\Sigma(S)$  (eg., if  $R_S = R_{S'}$ ) we examine the fixed-point sets  $\Sigma(S)^{\rho(s)}$  and  $\Sigma(S)^{\rho(t)}$ . If they intersect, then  $m'(\rho(s), \rho(t)) < \infty$ .

Similarly,  $m(s, t) = \infty$  if and only if

$$\Sigma(S)^s \cap \Sigma(S)^t = \Sigma^{W_{\{s,t\}}} = \emptyset.$$

When  $R_S = R_{S'}$ , it is necessary that

$$\Sigma(S)^{\rho(s)} \cap \Sigma(S)^{\rho(t)} = \Sigma(S)^{W_{\{\rho(s), \rho(t)\}}} = \emptyset$$

to ensure that  $m'(\rho(s), \rho(t)) = \infty$ . It is therefore essential to understand the interaction of the walls corresponding to the elements of  $R_{S'}$  within the Davis complex  $\Sigma(S)$ .

## 5 Rigidity in Coxeter Groups of Type $K_n$

### 5.1 Coxeter Groups of Type $K_n$

It is natural to consider the consequences of relaxing the hypotheses of Theorem 4.2.2. With this in mind, we define a Coxeter system  $(W, S, m)$  to be of *type  $K_n$*  if

1. the Coxeter graph  $\Gamma_S$  is the complete graph on  $n$  vertices;
2. for all  $s, t \in S$ ,  $m(s, t)$  is odd.

The structure of Coxeter systems of type  $K_n$  imposes a structure on the nerves of such systems, as the following lemma shows.

**Lemma 5.1.1** *If  $(W, S)$  is a Coxeter system of type  $K_n$ , then the nerve  $N(S)$  is the complete graph on  $n$  vertices.*

Proof: Recall that the vertex set of  $N(S)$  is equal to  $S$ , so  $N(S)$  has exactly  $n$  vertices. If  $s, t$  are vertices of  $N(S)$ , then  $m(s, t) < \infty$  and so  $W_{\{s, t\}}$  is a finite subgroup of  $W$ . We conclude that the vertices  $s$  and  $t$  are joined by the edge  $\{s, t\}$  in  $N(S)$ . If  $a, b, c \in S$ , then  $a, b, c$  are the vertices of a cycle in  $\Gamma_S$  (since the graph  $\Gamma_S$  is complete). By Corollary 3.3.4, the subgroup  $W_{\{a, b, c\}}$  is infinite, so the set  $\{a, b, c\}$  cannot be a 2-simplex in  $N(S)$ . Therefore  $N(S)$  is a graph with  $n$  vertices with the property that each pair of vertices is joined by an edge. In other words, the nerve  $N(S)$  is the complete graph on  $n$  vertices.  $\square$

With Lemma 5.1.1 in hand we note the similarity between Coxeter systems of type  $HM_2$  and Coxeter systems of type  $K_n$ . In each case, the (modified) Coxeter graph is a complete graph (cf. Section 4.2). If  $(W, S)$  is a Coxeter

system of type  $HM_2$ , then for any vertex  $v$  in the modified Coxeter graph  $\tilde{\Gamma}_S$ , all but two of the edges emanating from  $v$  in  $\tilde{\Gamma}_S$  have infinite label (as in the discussion on page 39). If  $(W, S)$  is a type  $K_n$  Coxeter system and  $v$  is a vertex in the Coxeter graph  $\Gamma_S = \tilde{\Gamma}_S$ , then every edge emanating from  $v$  has finite label. We require that every edge label is odd in order to ensure that every element in  $W$  of order two is a reflection (cf. Lemma 5.4.4).

This chapter will be devoted to showing that if  $(W, S)$  is of type  $K_n$  and  $(W, S')$  is any other Coxeter system, then  $\Gamma_S$  is isomorphic to  $\Gamma_{S'}$ . In certain cases, this is sufficient to conclude that  $W$  is rigid.

## 5.2 Conjugacy of Maximal Special Subgroups

If  $G$  is a group and  $H$  is a subgroup of  $G$ , then  $H$  is said to be *maximal* if for every subgroup  $K$  of  $G$  with

$$H \subseteq K \subseteq G,$$

either  $K = G$  or  $K = H$ .

Let  $(W, S)$  be a Coxeter system. Recall that  $W\mathcal{S}(S)$  is the poset of all cosets of special subgroups in  $W$  (with a partial ordering of set-theoretic inclusion). If  $G$  is a finite subgroup of  $W$ , let  $F(G)$  be the subposet of  $W\mathcal{S}(S)$  fixed by  $G$ . Specifically,

$$\begin{aligned} F(G) &= \{wW_T \in W\mathcal{S}(S) : G \cdot wW_T = wW_T\} \\ &= \{wW_T \in W\mathcal{S}(S) : G \subseteq wW_Tw^{-1}\}. \end{aligned}$$

**Lemma 5.2.1** *If  $G$  is a maximal finite subgroup of  $W$ , then  $G \in P(S)$  (i.e.,  $G$  is an  $S$ -parabolic subgroup of  $W$ ).*

Proof: Since  $G$  is finite and  $\Sigma$  is a  $CAT(0)$  space, Theorem 2.4.3 states that the fixed-point set of the  $G$ -action on the Davis complex  $\Sigma$  is nonempty. By Theorem 3.4.2, the action of  $G$  on  $\Sigma$  is simplicial and, by Lemma 3.1.7, the  $G$ -action is free on the set of chambers of  $\Sigma$ . We deduce that  $G$  must fix a vertex  $v$  of  $\Sigma$  (i.e.  $v = wW_T$  for some  $w \in W$  and  $T \in \mathcal{S}(S)$ ). It follows that  $G$  is contained in the parabolic subgroup  $wW_Tw^{-1}$  and, since  $G$  is maximal,  $G = wW_Tw^{-1}$ .  $\square$

The following result provides a characterization of the subposets of the collection of cosets in  $W\mathcal{S}(S)$  fixed by special subgroups of  $W$ .

**Theorem 5.2.2** (Charney-Davis [11]) *Let  $A \in N(S)$  be a simplex in the nerve and let  $wW_T \in F(W_A)$  (i.e.,  $wW_T$  is a coset fixed by the action of  $W_A$  on the poset  $W\mathcal{S}(S)$ ), then*

- a) *the subposet  $F(W_A)_{wW_T \geq}$  is isomorphic to the poset of faces of a convex cell of dimension  $|T| - |A|$ ;*
- b) *the subposet  $F(W_T)_{wW_T <}$  is isomorphic to  $\mathcal{S}(S)_{T <}$*

Inherent in the structure of Coxeter groups of type  $K_n$  (as well as type  $HM_n$  and right-angled groups) is the fact that all maximal parabolic subgroups have equal rank. In other words, each maximal simplex in the nerve has the same dimension. It is also apparent that the nature of the collection of parabolic subgroups is intrinsic to the topology of the Davis complex: each simplex in  $\Sigma$  corresponds to a chain of cosets

$$w_1W_{T_1} < w_2W_{T_2} < \dots < w_kW_{T_k}$$

which in turn determines a sequence of parabolic subgroups

$$w_1W_{T_1}w_1^{-1}, w_2W_{T_2}w_2^{-1}, \dots, w_kW_{T_k}w_k^{-1}.$$

It is then natural to investigate the conjugacy classes of maximal parabolic subgroups. With these thoughts in mind, let  $k$  be a fixed positive integer. A Coxeter system  $(W, S)$  has *condition  $k$*  if for every maximal simplex  $A \in N^*(S)$ , the set  $A$  satisfies  $|A| = k$ . Note that this is equivalent to the statement that for every maximal simplex  $A \in N^*(S)$  in the nerve  $N(S)$ , the special subgroup  $W_A$  has parabolic rank  $k$ .

**Lemma 5.2.3** *Assume that the Coxeter system  $(W, S)$  has condition  $k$ . If  $G$  is a maximal finite subgroup of  $W$ , then  $rk(G) = k$ .*

Proof: By Lemma 5.2.1,  $G$  is a parabolic subgroup of  $W$ , so  $G = wW_Tw^{-1}$  for some  $w \in W$  and  $T \in \mathcal{S}(S)$ . If  $T \in N^*(S)$ , then we are done. Otherwise,  $T$  must be properly contained in some maximal simplex  $A \in N^*(S)$ . If this is the case, then  $G$  is properly contained in the finite parabolic subgroup  $wW_Aw^{-1}$ , contradicting the maximality of  $G$ .  $\square$

**Lemma 5.2.4** *Assume that the Coxeter system  $(W, S)$  has condition  $k$ . If  $A \in N^*(S)$  is a maximal simplex in the nerve, then  $W_A$  is a maximal subgroup of  $W$ . In particular, the set*

$$F(W_A) = \{wW_T \in WS(S) : W_A = wW_Tw^{-1}\}$$

*is equal to the conjugacy class of  $W_A$  in  $W$ .*

Proof: Let  $G$  be a maximal finite subgroup of  $W$  with  $W_A \subseteq G$ . By Lemma 5.2.1,  $G$  is a parabolic subgroup of  $W$  and, by Lemma 5.2.3,  $rk(G) = rk(W_A) = k$ . Therefore, by Lemma 4.1.2 c,  $G = W_A$ .  $\square$

As indicated, we intend to shed light on the nature of the conjugacy classes of maximal parabolic subgroups in Coxeter systems which have condition  $k$ .

To do so we need only consider the conjugacy classes corresponding to special subgroups of the form  $W_A$ , where  $A \in N^*(S)$  is a maximal simplex in the nerve. With Lemma 5.2.4 in hand this is equivalent to investigating the fixed poset  $F(W_A)$ . The next result shows that, given an arbitrary Coxeter system  $(W, S)$  and a maximal simplex  $A \in N^*(S)$  such that  $W_A$  is a maximal subgroup (such a situation occurs when a Coxeter system has condition  $k$ ) the composition of  $F(W_A)$  is quite simple.

**Lemma 5.2.5** *Let  $(W, S)$  be an arbitrary Coxeter system and let  $A \in N^*(S)$  be a maximal simplex in the nerve. In addition, suppose that  $W_A$  is a maximal finite subgroup of  $W$ . If  $wW_T \in F(W_A)$ , then  $F(W_A)_{wW_T \geq}$  is a singleton.*

Proof: By Theorem 5.2.2 a, the subposet  $F(W_A)_{wW_T \geq}$  is isomorphic to the poset of faces of a convex cell of dimension  $|T| - |A|$ . Since  $wW_T \in F(W_A)_{wW_T \geq}$ , the subposet  $F(W_A)_{wW_T \geq}$  is nonempty. We conclude that  $|T| - |A| \geq 0$  and therefore  $|T| \geq |A|$ . Since  $W_A = wW_T w^{-1}$ , it follows that  $w^{-1}W_A \in F(W_T)$  and, once again by Theorem 5.2.2 a,  $|A| \geq |T|$ , implying that  $|A| = |T|$ . Therefore the dimension of the aforementioned cell is zero and the result follows.  $\square$

The following observation is crucial to the subsequent development of the theory of type  $K_n$  Coxeter groups.

**Theorem 5.2.6** *Let  $(W, S)$  be an arbitrary Coxeter system and suppose that  $A$  and  $B$  are distinct simplices in the nerve  $N(S)$ . If  $W_A$  and  $W_B$  are maximal special subgroups of  $W$ , then  $W_A$  and  $W_B$  are not conjugate in  $W$ .*

Proof: Since  $W_A$  is a maximal finite subgroup of  $W$ , the fixed poset of  $W_A$  acting on  $W\mathcal{S}(S)$  is given by

$$F(W_A) = \{wW_T : W_A = wW_T w^{-1}\}.$$

It suffices to show that  $F(W_A) - W_A = \emptyset$ . To this end we suppose the contrary. Let  $wW_T \in F(W_A)$  with  $wW_T \neq W_A$ . By Lemma 5.2.5,  $F(W_A)_{wW_T \geq}$  is a singleton. It follows that  $\Sigma^{W_A}$  is a collection of 0-cells and, in particular, contains no non-constant path. Since  $W$  acts by isometry on the Davis complex  $\Sigma$ , the (unique) geodesic segment  $[W_A, wW_T]$  is contained in the wall  $\Sigma^\alpha$  for each  $\alpha \in A$ . This implies that

$$[W_A, wW_T] \subseteq \bigcap_{\alpha \in A} \Sigma^\alpha = \Sigma^{W_A},$$

which is impossible.  $\square$

**Corollary 5.2.7** *Assume that the Coxeter system  $(W, S)$  has condition  $k$ . If  $(W, S')$  is another Coxeter system for  $W$ , then there exists a bijection*

$$\phi : N^*(S) \longrightarrow N^*(S')$$

*of maximal simplices of the associated nerves.*

Proof: Assume that  $A \in N^*(S)$  is a maximal simplex in the nerve  $N(S)$ . By Lemma 5.2.4, the special subgroup  $W_A$  is a maximal finite subgroup of  $W$ . Applying Lemma 5.2.1, we see that  $W_A$  is a  $S'$ -parabolic subgroup, so there is a  $w \in W$  and  $A' \in \mathcal{S}(S')$  such that  $W_A = wW_{A'}w^{-1}$ . Theorem 5.2.6 implies that  $A'$  is unique. Since  $W_{A'}$  is a maximal subgroup of  $W$ , it follows that  $A'$  is a maximal simplex in the nerve  $N^*(S')$ . By letting  $\phi(A) = A'$ , we have  $W_A = wW_{\phi(A)}w^{-1}$ .  $\square$

### 5.3 The Schur Multiplier of a Coxeter Group

In [21], R. Howlett provides a remarkable algorithm for computing the second cohomology of a Coxeter group  $W$  with system  $(W, S)$ . This algorithm relies only on the isomorphism-type and labeling scheme for the graph  $\Gamma_S$ .

Let  $G$  be a group and let  $\mathbb{C}^\times$  be the multiplicative group of complex numbers with trivial  $G$ -action. The *Schur multiplier*,  $Mult(G)$ , of  $G$  is defined as

$$Mult(G) = H^2(G, \mathbb{C}^\times).$$

If  $(W, S, m)$  is a Coxeter system, let

$$A_2 = \{\{s, t\} \subseteq S : m(s, t) = 2\}.$$

Write  $\{s, t\} \approx \{s, t'\}$  if  $\{s, t\}, \{s, t'\} \in A_2$  and  $m(t, t')$  is odd. Let  $\sim$  be the equivalence relation on  $A_2$  generated by  $\approx$ . Let  $\Gamma'_S$  be the graph obtained by deleting from  $\Gamma_S$  the edges with even label. Define

$\mu_S =$  the number of edges of  $\Gamma_S$  with finite edge label;

$\nu_S =$  the number of equivalence classes of  $\sim$  on  $A_2$ ;

$\zeta_S =$  the number of connected components of  $\Gamma'_S$ .

**Theorem 5.3.1** (R. Howlett [21]) *If  $(W, S)$  is any Coxeter system for  $W$ , then*

$$Mult(W) \cong \mathbb{Z}_2^{\mu_S + \nu_S + \zeta_S - |S|}.$$

S. Pride and R. Stöhr ([28], Corollary, p. 62) prove a similar result in the case that  $(W, S)$  is “aspherical” (i.e., any three distinct elements of  $S$  generate an infinite subgroup of  $W$ ).

## 5.4 Coxeter Graphs for Groups of Type $K_n$

Unless otherwise stated, we assume throughout this section that  $(W, S, m)$  is a Coxeter system of type  $K_n$ . In this case we have

1. the nerve  $N(S)$  is the complete graph on  $n = |S|$  vertices (note that this implies that the Coxeter system  $(W, S)$  has condition 2 (cf. Section 5.2);
2. each edge  $\{s, t\}$  in the Coxeter graph  $\Gamma_S$  corresponds to a maximal simplex  $\{s, t\}$  in the nerve  $N(S)$ ;
3. for each edge  $\{s, t\}$  in the Coxeter graph  $\Gamma_S$ , the corresponding special subgroup  $W_{\{s,t\}}$  is a maximal finite subgroup of  $W$  and is isomorphic to a dihedral group of order  $2m(s, t)$ ;
4. if  $n \geq 3$ , then for any  $s \in S$ , the special subgroup  $W_{\{s\}}$  can be expressed

as

$$W_{\{s\}} = \bigcap_{i=1}^{n-1} W_{A_i},$$

where  $\{A_i\}_{i=0}^{n-1}$  is the collection of edges in  $\Gamma_S$  containing the vertex  $s$  (this is a direct consequence of Theorem 3.2.5).

Given a Coxeter system  $(W, S)$ , recall that for each positive integer  $k$ , the set  $P_k(S)$  is the collection of  $S$ -parabolic subgroups of  $W$  which have parabolic rank  $k$ . The following result is proved in [11].

**Theorem 5.4.1** (R. Charney, M. Davis, [11]) *Suppose  $(W, S)$  and  $(W, S')$  are Coxeter systems for any Coxeter group  $W$ . If  $P_1(S) = P_1(S')$ , then  $P_k(S) = P_k(S')$  for all  $k \geq 1$ .*

We use this to prove the following analog of Theorem 4.2.5.

**Theorem 5.4.2** *Let  $(W, S)$  be a Coxeter system of type  $K_n$  and let  $(W, S')$  be another Coxeter system for  $W$ . For all  $k \geq 1$ ,  $P_k(S) = P_k(S')$ .*

Proof: By Theorem 5.4.1, it suffices to show that  $P_1(S) = P_1(S')$ . We consider three cases.

Case 1. If  $n = 1$ , then the subgroup lattice of  $W$  consists of the trivial subgroup and  $W$  itself. Thus  $P_1(S) = W = P_1(S')$ .

Case 2. If  $n = 2$ , then  $W$  is isomorphic to the dihedral group  $D_{2k+1}$  for some  $k \geq 1$  and  $W$  is the full symmetry group of a regular  $(2k + 1)$ -gon  $P$  in  $\mathbb{R}^2$ . Since  $S = \{s_1, s_2\}$  generates  $W$ , and each element of  $S$  has order 2, there are lines of symmetry  $L_1$  and  $L_2$  through  $P$  such that  $s_1$  (resp.  $s_2$ ) acts on  $P$  by reflection through the line  $L_1$  (resp.  $L_2$ ) (every symmetry of  $P$  with order 2 is a reflection because  $P$  has an odd number of sides), and  $L_1$  and  $L_2$  form an angle of  $\frac{q\pi}{2k+1}$  where  $q$  and  $2k + 1$  are relatively prime. Note that each line of symmetry through  $P$  corresponds uniquely to a reflection  $r \in R_S$ . Since each  $s' \in S'$  has order 2,  $s'$  acts on  $P$  by reflection through a line of symmetry  $L_{s'}$  and is therefore conjugate in  $W$  to an element of  $S$ . Let  $G \in P_1(S)$ . Then  $G = wW_{\{s\}}w^{-1}$  for some  $s \in S$  and  $w \in W$ . Choosing any  $s' \in S'$  we see that there is an element  $v \in W$  such that  $s = vs'v^{-1}$ . It follows that

$$G = wW_{\{s\}}w^{-1} = \{1, wsw^{-1}\} = \{1, wvs'v^{-1}w^{-1}\} = wvW_{\{s'\}}(wv)^{-1}.$$

Consequently  $G \in P_1(S')$  and so  $P_1(S) \subseteq P_1(S')$ . The same argument shows that  $P_1(S') \subseteq P_1(S)$ .

Case 3. Assume  $n \geq 3$ . Choose  $G \in P_1(S)$ . Then  $G = wW_{\{s\}}w^{-1}$  for some  $s \in S$  and  $w \in W$ . Letting  $\{A_i\}_{i=1}^{n-1}$  be the collection of edges of  $\Gamma_S$  containing  $s$ , we have

$$\begin{aligned} G &= w \left( \bigcap_{i=1}^{n-1} W_{A_i} \right) w^{-1} \\ &= \bigcap_{i=1}^{n-1} (wW_{A_i}w^{-1}). \end{aligned}$$

By Corollary 5.2.7, we know that for each  $1 \leq i \leq n - 1$ , the subgroup  $wW_{A_i}w^{-1}$  is an  $S'$ -parabolic subgroup (i.e.,  $wW_{A_i}w^{-1} \in P(S')$ ) and, by Lemma 4.1.2 a, the intersection of parabolic subgroups is a parabolic subgroup. Therefore  $G \in P_1(S')$ , implying that  $P_1(S) \subseteq P_1(S')$ .

Now choose  $G \in P_1(S')$ . Then  $G = vW_{\{t\}}v^{-1}$  for some  $v \in W$  and  $t \in S'$ . Let  $B \in N^*(S')$  be a maximal simplex in the nerve  $N(S')$  such that  $W_{\{t\}} \subseteq W_B$  (the vertex  $t \in N(S')$  is contained in some maximal simplex  $B \in N^*(S')$  in the nerve  $N(S')$ ). By Corollary 5.2.7,  $W_B = uW_Au^{-1}$  for some  $u \in W$  (where  $A = \phi^{-1}(B) \in N^*(S)$  is a maximal simplex in the nerve  $N(S)$ ). The parabolic subgroup  $uW_Au^{-1}$  is isomorphic to a dihedral group  $D_{2k+1}$ , for some positive integer  $k$ . Also,  $v^{-1}tv \in uWu^{-1}$  and has order 2, implying that  $v^{-1}tv \in R_S$  (any element in  $D_{2k+1}$  of order 2 is a reflection). Hence  $t \in R_S$ . This implies that  $W_{\{t\}} \in P_1(S)$  and therefore  $G \in P_1(S)$ . We now have  $P_1(S') \subseteq P_1(S)$  which completes the proof in the case that  $n \geq 3$ .

Thus for every positive integer  $n$  we have the desired result.  $\square$

As a consequence we have:

**Corollary 5.4.3** *Let  $(W, S)$  be a Coxeter system of type  $K_n$  and let  $(W, S')$  be any other Coxeter system for  $W$ . Let  $\phi : N^*(S) \rightarrow N^*(S')$  be the bijection as in Corollary 5.2.7. If  $\{a, b\}$  is an edge in  $N(S)$ , then  $\phi(\{a, b\})$  is an edge in  $N(S')$ .*

Proof: The edge  $\{a, b\}$  corresponds to the  $S$ -parabolic subgroup  $W_{\{a, b\}}$ , which has parabolic rank 2. By Corollary 5.2.7,  $W_{\{a, b\}} = wW_{\phi(\{a, b\})}w^{-1}$  for some  $w \in W$  and, by Theorem 5.4.2,  $wW_{\phi(\{a, b\})}w^{-1}$  has parabolic rank 2 as well. In other words,  $\phi(\{a, b\})$  is an edge in  $N(S')$ .  $\square$

Though Corollary 5.2.7 gave us a bijection between the maximal simplices of the nerves, prior to Theorem 5.4.2 we were unable to determine whether the bijection was dimension preserving.

Given a Coxeter system  $(W, S)$  of type  $K_n$  and any other Coxeter system  $(W, S')$  for  $W$ , we now intend to show that the Coxeter graphs  $\Gamma_S$  and  $\Gamma_{S'}$  are isomorphic (as simplicial complexes). This is accomplished by establishing that the number of vertices and edges of  $\Gamma_{S'}$  are the same as those of  $\Gamma_S$ .

**Lemma 5.4.4** *Let  $(W, S, m)$  be a Coxeter system of type  $K_n$ . If  $(W, S', m')$  is any other Coxeter system for  $W$  and  $\{p, q\} \in \mathcal{S}(S')$ , then  $m'(p, q)$  is odd.*

Proof: Suppose  $m'(p, q)$  is even for some  $p, q \in S'$ . In this case,  $W_{\{p, q\}}$  cannot be a maximal subgroup of  $W$  (every maximal finite subgroup of  $W$  is isomorphic to a dihedral group  $D_{2k+1}$  for some integer  $k \geq 0$ ). By Lemma 4.1.2 c, if  $B \in \mathcal{S}(S')$  is such that  $W_{\{p, q\}}$  is properly contained in  $W_B$ , then  $|B| \geq 3$ . By Theorem 5.4.2,  $P_k(S') = P_k(S) = \emptyset$  for  $k \geq 3$ . Thus  $m'(p, q)$  must be odd, contradicting our assumption that  $m'(p, q)$  is even..  $\square$

Recall that if  $(W, S, m)$  is a Coxeter system for  $W$ , then  $\mu_S$  is the number of edges in the Coxeter graph with finite edge label.

**Lemma 5.4.5** *Let  $(W, S, m)$  be a Coxeter system of type  $K_n$ . If  $(W, S', m')$  is another Coxeter system for  $W$ , then  $\mu_S = \mu_{S'}$ .*

Proof: Let  $\{a, b\}$  be an edge in  $\Gamma_S$ . Since  $m(a, b)$  is odd (and therefore finite),  $\{a, b\}$  is also an edge in  $N^*(S)$ . By Corollary 5.2.7 and Corollary 5.4.3, the edge  $\{a, b\}$  corresponds uniquely to an edge  $\phi(\{a, b\})$  in  $N^*(S')$  for some  $p, q \in S'$  (where  $\phi$  is the bijection of Corollary 5.2.7), so  $\phi(\{a, b\})$  is an edge in  $\Gamma_{S'}$ . Therefore  $\mu_{S'} \geq \mu_S$ . If  $\{p, q\}$  is an edge in  $\Gamma_{S'}$  with  $m'(p, q) \neq \infty$ , the previous argument reverses, showing that  $\mu_{S'} \leq \mu_S$ .  $\square$

**Theorem 5.4.6** *Let  $(W, S, m)$  be a Coxeter system of type  $K_n$ . If  $(W, S', m')$  is another Coxeter system for  $W$ , then  $|S| = |S'|$ .*

Proof: By Lemma 5.4.4,  $m'(s, t)$  is odd for all  $s, t \in \mathcal{S}(S')$ . From this we conclude that  $\nu_{S'} = 0$  and  $\zeta_{S'} = 1$  (there are no even edge labels). By Theorem 5.3.1,

$$\mathbb{Z}_2^{\mu_S+1-|S|} \cong \text{Mult}(W) \cong \mathbb{Z}_2^{\mu_{S'}+1-|S'|}.$$

Lemma 5.4.5 forces  $|S| = |S'|$ . □

From this we deduce:

**Corollary 5.4.7** *If  $(W, S, m)$  is a Coxeter system of type  $K_n$  and  $(W, S', m')$  is any other Coxeter system for  $W$ , then*

- a) *the Coxeter graphs  $\Gamma_S$  and  $\Gamma_{S'}$  are isomorphic (as simplicial complexes);*
- b) *the nerves  $N(S)$  and  $N(S')$  are isomorphic (as simplicial complexes);*
- c) *the set of edge labels appearing in  $\Gamma_S$  is precisely the set of edge labels appearing in  $\Gamma_{S'}$  (counting multiplicity).*

Proof: Coxeter graphs have the property that the endpoints of every edge are distinct (i.e., they have no loops). Up to isomorphism, there is only one such graph which has  $|S|$  vertices and  $\mu_S$  edges (the maximal number of edges possible). Thus  $\Gamma_S$  is isomorphic to  $\Gamma_{S'}$ . By Lemma 5.4.4, each edge label of  $\Gamma_{S'}$  is odd. This implies that  $(W, S')$  is a Coxeter system of type  $K_n$  as well.

By Lemma 5.1.1,  $N(S')$  is the complete graph on  $n$  vertices, so  $N(S)$  is isomorphic to  $N(S')$ .

The bijection  $\phi : N^*(S) \longrightarrow N^*(S')$  of Corollary 5.2.7 provides a well-defined one-to-one correspondence between the conjugacy classes of maximal

$S$ -special subgroups and maximal  $S'$ -special subgroups:  $[W_{\{s,t\}}] \mapsto [W_{\phi(\{s,t\})}]$  (where  $[W_{\{s,t\}}]$  denotes the conjugacy class of the subgroup  $W_{\{s,t\}}$  in  $W$ ). Thus each label  $m(s,t)$  appearing in  $\Gamma_S$  corresponds uniquely to the same edge label in  $\Gamma_{S'}$ .  $\square$

We emphasize that Corollary 5.4.7 provides an isomorphism in the simplicial complex sense (as in page 32). The graphs  $\Gamma_S$  and  $\Gamma_{S'}$  are not necessarily graph-isomorphic (as defined on page 25).

In light of the preceding discussion, if  $(W, S)$  is a Coxeter system of type  $K_n$ , then any other Coxeter system for  $W$  is also of type  $K_n$ ; we may therefore define a Coxeter *group* to be of *type*  $K_n$  if it has a Coxeter system of type  $K_n$ .

Enumerate  $S$  as

$$S = \{s_1, s_2, \dots, s_n\}.$$

For  $1 \leq i \leq n$ , let

$$S_i = \{s_1, \dots, \hat{s}_i, \dots, s_n\}.$$

Let

$$Diag(S) = \{(s_i, s_j) \in S \times S : i = j\}.$$

With this notation in hand we may state a rigidity result in a special case for Coxeter systems of type  $K_n$ , the proof of which follows immediately from Corollary 5.4.7.

**Corollary 5.4.8** *Let  $(W, S, m)$  be a Coxeter system of type  $K_n$ . Suppose that for some  $1 \leq i < j \leq n$ ,*

$$m|_{(S_i \times S_j - Diag(S))} : (S_i \times S_j - Diag(S)) \longrightarrow \mathbb{N}$$

*is constant. Then  $W$  is rigid.*

The second hypothesis is equivalent to the statement that all but one of the edge labels on  $\Gamma_S$  are the same.

Note that if  $W$  is of type  $K_2$ , then  $W$  is isomorphic to a dihedral group  $D_{2k+1}$  for some  $k \geq 1$  and satisfies the hypothesis of Corollary 5.4.8. As a consequence we have:

**Corollary 5.4.9** *If  $W$  is a Coxeter group which is isomorphic to a dihedral group  $D_{2k+1}$  for some  $k \geq 1$ , then  $W$  is rigid.*

Within the class of Coxeter graphs of type  $K_n$ , Corollary 5.4.8 provides an infinite subclass of type  $K_n$  Coxeter groups which are indeed rigid (namely type  $K_n$  groups such that all but one edge label differs from the others). For the remainder of this paper we discuss a strategy for extending rigidity to the entire class of type  $K_n$  Coxeter groups. This approach requires a deeper understanding of the Davis complex, and in particular the interaction of the walls contained within. While not providing a definitive answer to the rigidity question for type  $K_n$  Coxeter groups, this strategy yields both algebraic and geometric information and it is hoped that such an approach will open up avenues for exploration, especially in the geometric vein via the theory of  $CAT(0)$  groups.

## 6 Centralizers in Coxeter Groups of Type $K_n$

### 6.1 Extending Rigidity

From the discussion of Section 4.2, it is clear that the key to a rigidity result for Coxeter groups of type  $K_n$  is understanding the intersection of walls corresponding to reflections in the Davis complex. In this chapter, we consider a strategy which provides a partial understanding. In the spirit of geometric group theory, this strategy involves the examination of group actions on these walls.

As mentioned in Section 5.1, the Coxeter groups of type  $K_n$  are a natural generalization of the Coxeter groups of type  $HM_2$ . The type  $K_n$  Coxeter groups and their associated Davis complexes are more complicated than their type  $HM_2$  counterparts. For example, if  $r$  is a reflection in a type  $HM_2$  group, the wall  $\Sigma^r$  in the Davis complex  $\Sigma$  is a contractible homology 1-manifold (Theorems 2.4.4 and 4.2.3) and is therefore homeomorphic to the real line. The situation is quite different in the Davis complex corresponding to a type  $K_n$  group, as the following lemma illustrates.

**Lemma 6.1.1** *Let  $(W, S)$  be a Coxeter system of type  $K_n$ . If  $r$  is a reflection in  $W$ , then the wall  $\Sigma^r$  is isomorphic to the barycentric subdivision of the  $(n-1)$ -valent tree (the  $(n-1)$ -valent tree is the simply connected graph with  $n-1$  edges emanating from each vertex).*

Proof: If  $r \in W$  is a reflection, then  $r$  is conjugate in  $W$  to some generator  $s \in S$  (i.e.  $r = vsv^{-1}$ ). This implies that  $\Sigma^r = v\Sigma^s$ , so we may assume without loss of generality that  $r \in S$  and determine the structure of the walls corresponding to generators. Since  $(W, S)$  is a type  $K_n$  Coxeter system, we observe that if

$s, t \in S$  are generators, then  $s$  is conjugate in  $W$  to  $t$ . Therefore every wall in the Davis complex  $\Sigma$  is isomorphic to the wall  $\Sigma^r$  for some fixed  $r \in S$  and we need only determine the structure of a single wall  $\Sigma^r$ . Since the poset  $W\mathcal{S}(S)$  of cosets of special subgroups of  $W$  contains no chains of length three or greater in a Coxeter system of type  $K_n$ , the wall  $\Sigma^r$  must be a graph. By Theorem 2.4.4,  $\Sigma^r$  is contractible and therefore must be a tree. If  $s \in S$  is a generator,  $w \in W$ , and  $v = wW_{\{s\}}$  is a vertex in  $\Sigma^r$  (i.e.,  $r = wsw^{-1}$ ), then Theorem 5.2.2 b implies that there are  $n - 1$  edges emanating from  $v$  (each edge corresponding to a chain  $wW_{\{s\}} < uW_T$  in  $W\mathcal{S}(S)$ , where  $uW_T$  is a coset containing  $wW_{\{s\}}$  and  $uW_T$  is a vertex in  $\Sigma^r$ ). If  $yW_T$  is a vertex in  $\Sigma^r$  with  $|T| = 2$ , then Theorem 5.2.2 a implies that  $yW_T$  has valence two. The edges which emanate from  $yW_T$  are precisely the edges with opposite endpoints having valence  $n - 1$ .  $\square$

This example underscores the difficulty in proving (or disproving) that a given Coxeter group of type  $K_n$  is rigid.

## 6.2 Elements of Minimal and Maximal Length

In this section, we collect some standard facts about Coxeter systems which are essential to the material which follows. Let  $(W, S)$  be an arbitrary Coxeter system. Given  $T \in \mathcal{S}(S)$  and  $w \in W$ , define

$$A_T = \{w \in W : l(wt) > l(w) \text{ for all } t \in T\},$$

and

$$S(w) = \{s \in S : l(ws) < l(w)\}.$$

**Lemma 6.2.1** ([6], Exercise 3, p. 37) *Given  $w \in W$  and  $T \subseteq S$ , there is a unique element  $v_T$  of shortest length in the coset  $wW_T$ . Moreover, the following statements are equivalent:*

- a)  $v_T = w$ ;
- b)  $w \in A_T$ ;
- c) for each  $u \in W_T$ ,  $l(wu) = l(w) + l(u)$ .

**Lemma 6.2.2** ([6], Exercise 22, p. 43) *If  $T \in \mathcal{S}(S)$ , then there is a unique element  $w_T \in W_T$  of longest length.*

**Lemma 6.2.3** (M. Davis, [14], Lemma 1.6) *If  $T \in \mathcal{S}(S)$  and  $w \in W$ , then there exists a unique element in  $wW_T$  of longest length (namely, the element  $v_T w_T$ ). Moreover, the following are equivalent:*

- a)  $w$  is the element of longest length in  $wW_T$ ;
- b)  $w = uw_T$ , for some  $u \in A_T$ ;
- c)  $T \subseteq S(w)$ .

**Lemma 6.2.4** *Let  $s, t \in S$  be such that  $m(s, t)$  odd. If  $z = (st)^{\frac{m(s,t)-1}{2}}$ , then  $tz$  is the unique element of maximal length in  $W_{\{s,t\}}$ .*

Proof: Observe that

$$tz = tst \cdots st = sts \cdots ts$$

where the two latter expressions are of length  $m(s, t)$ . Lemma 3.2.1 e implies that

$$l((tz)t) = l(tz) - 1,$$

and

$$l((tz)s) = l(tz) - 1.$$

From this we see that  $\{s, t\} \subseteq S(tz)$ . By Lemma 6.2.3 c, it follows that  $tz$  is the unique element of longest length in  $tzW_{\{s,t\}} = W_{\{s,t\}}$ .  $\square$

### 6.3 The Action of $C_W(r)$ on $\Sigma^r$

Given a group  $G$  and an element  $g \in G$ , the *centralizer*  $C_G(g)$  of  $g$  in  $G$  is the set of all elements in  $G$  that commute with  $g$ . That is

$$C_G(g) = \{x \in G : xg = gx\}.$$

One can easily show that  $C_G(g)$  is a subgroup of  $G$ .

**Lemma 6.3.1** *Given a Coxeter system  $(W, S)$  and  $r \in S$ , the action of  $W$  on the Davis complex  $\Sigma$  induces an action of the centralizer  $C_W(r)$  on the wall  $\Sigma^r$ .*

Proof: Since the action of  $W$  on the Davis complex  $\Sigma$  is simplicial, we need only show that if  $x \in C_W(r)$  and  $v$  is a vertex in  $\Sigma^r$ , then  $x \cdot v$  is a vertex in  $\Sigma^r$  as well. Recall that  $v$  is a vertex of the wall  $\Sigma^r$  if and only if  $v = wW_T$  for some  $w \in W$  and  $T \in \mathcal{S}(S)$  such that  $r \in wW_Tw^{-1}$ . If  $x \in C_W(r)$ , then

$$x^{-1}rx = r \in wW_Tw^{-1},$$

implying that  $r \in xwW_Tw^{-1}x^{-1}$ . From this we deduce that  $x \cdot v = xwW_T$  is a vertex of  $\Sigma^r$ .  $\square$

Given a Coxeter system  $(W, S)$  and  $r \in R_S$  (recall that  $R_S$  is the set of reflections in  $W$ ), B. Brink [9] has shown that the centralizer  $C_W(r)$  is the semidirect product of a Coxeter group by a free group. We prove an analogous result in the case that  $W$  is a Coxeter group of type  $K_n$  by a different method. In the process we develop a technique for characterizing the collection of mirrors which meet  $\Sigma^r$ . This is the first step required in alleviating the problem discussed in Section 6.1.

For  $r, s \in S$ , let

$$F_{r,s} = \{w \in W : w\Sigma^r = \Sigma^s \text{ and } l(sw) > l(w)\}.$$

In other words  $F_{r,s}$  is the set of  $w \in W$  which carry  $\Sigma^r$  to  $\Sigma^s$  such that left multiplication of  $w$  by  $s$  increases length.

**Lemma 6.3.2** *Let  $r \in S$ . If  $u \in F_{r,r}$ , then  $K$  and  $uK$  lie in the same component of  $\Sigma - \Sigma^r$  (where  $K$  is the fundamental chamber in the Davis complex defined on page 34).*

Proof: Suppose  $\Sigma^r$  separates  $K$  and  $uK$ . It follows that  $ruK$  and  $K$  are on the same side of  $\Sigma^r$ . If  $u = u_1u_2 \cdots u_k$  is a reduced expression for  $u$ , then the gallery

$$K, u_1K, u_1u_2K, \dots, u_1u_2 \cdots u_kK$$

crosses  $\Sigma^r$  exactly once (by Lemma 3.1.8). This implies that the gallery

$$K, u_1K, \dots, u_1 \cdots u_kK, ru_1, \dots, u_kK$$

crosses  $\Sigma^r$  exactly twice. By Lemma 3.1.8 and Lemma 3.2.1 e,

$$l(ru) = k - 1 < l(u) = k,$$

which is a contradiction. □

**Theorem 6.3.3** *If  $r \in S$ , then  $F_{r,r}$  is a subgroup of the centralizer  $C_W(r)$ .*

Proof: If  $x \in F_{r,r}$ , then

$$x\Sigma^r = \Sigma^{xrx^{-1}} = \Sigma^r.$$

By Lemma 3.1.1, it follows that  $xrx^{-1} = r$ . Consequently  $x \in C_W(r)$  and therefore  $F_{r,r} \subseteq C_W(r)$ . We must show that  $F_{r,r}$  is closed under multiplication and inversion.

Pick  $u, v \in F_{r,r}$ . Since  $r$  commutes with  $u$  we apply Lemma 3.2.1 a to obtain:

$$l(ru^{-1}) = l((ru^{-1})^{-1}) = l(ur) = l(ru) > l(u).$$

Since  $u^{-1}\Sigma^r = \Sigma^r$ , it follows that  $u^{-1} \in F_{r,r}$ , implying that  $F_{r,r}$  is closed under inversion.

Applying  $uv$  to the Davis complex  $\Sigma$  we have  $K$  and  $uvK$  in the same component of  $\Sigma - \Sigma^r$  (by Lemma 6.3.2), which implies that  $\Sigma^r$  separates  $K$  and  $ruvK$ . If  $uv = s_1s_2 \cdots s_k$  is a reduced expression for  $uv$ , then the gallery

$$K, s_1K, s_1s_2K, \dots, s_1, s_2 \cdots s_kK$$

never crosses  $\Sigma^r$  (by Lemma 3.1.8), and to reach the chamber  $ruvK$  we must cross through  $\Sigma^r$ . By Lemma 3.1.8 and Lemma 3.2.1 e,

$$l(ruv) = l(uv) + 1 > l(uv).$$

Since

$$uv\Sigma^r = u(v\Sigma^r) = u\Sigma^r = \Sigma^r$$

it follows that  $uv \in F_{r,r}$ . Therefore  $F_{r,r}$  is closed under multiplication, completing the proof.  $\square$

**Theorem 6.3.4** *If  $r \in S$ , then the centralizer  $C_W(r)$  decomposes as the direct product  $C_W(r) = W_{\{r\}} \times F_{r,r}$ .*

Proof: We proceed in three steps.

Claim 1: The intersection  $W_{\{r\}} \cap F_{r,r}$  is trivial.

Observing that

$$0 = l(r \cdot r) < l(r) = 1,$$

it follows that  $r \notin F_{r,r}$ . Since  $W_{\{r\}} = \{1, r\}$ , we conclude that

$$W_{\{r\}} \cap F_{r,r} = \{1\}.$$

Claim 2: The centralizer  $C_W(r) = W_{\{r\}} \cdot F_{r,r}$ .

Let  $x \in W_{\{r\}} \cdot F_{r,r}$ . Then  $x = yz$  for some  $y \in W_{\{r\}}$  and  $z \in F_{r,r}$ . Then

$$\Sigma^{xrx^{-1}} = x\Sigma^r = yz\Sigma^r = y\Sigma^r = \Sigma^r.$$

By Lemma 3.1.1, we see that  $xrx^{-1} = r$ . This implies that  $x \in C_W(r)$  and so  $W_{\{r\}} \cdot F_{r,r} \subseteq C_W(r)$ . It remains to show the reverse inclusion. To this end, let  $x \in C_W(r)$ . Since  $x\Sigma^r = \Sigma^r$ , if  $l(rx) > l(x)$ , then

$$x \in F_{r,r} \subseteq W_{\{r\}} \cdot F_{r,r}.$$

Suppose  $l(rx) < l(x)$ . Then

$$rx\Sigma^r = xr\Sigma^r = x\Sigma^{r^3} = x\Sigma^r.$$

Noting that

$$l(r \cdot rx) = l(x) > l(rx)$$

we deduce that  $rx \in F_{r,r}$  and so

$$x = r \cdot rx \in W_{\{r\}} \cdot F_{r,r}.$$

This implies that  $C_W(r) \subseteq W_{\{r\}} \cdot F_{r,r}$ , hence  $C_W(r) = W_{\{r\}} \cdot F_{r,r}$  as desired.

Claim 3: The subgroups  $W_{\{r\}}$  and  $F_{r,r}$  are normal subgroups of the centralizer  $C_W(r)$ .

Let  $x \in C_W(r)$ . Since  $x$  commutes with  $r$  we have

$$xW_{\{r\}} = \{x, xr\} = \{x, rx\} = W_{\{r\}}x.$$

Therefore  $W_{\{r\}}$  is normal in  $C_W(r)$ . By Claim 2, the subgroup  $F_{r,r}$  has index 2 in  $C_W(r)$ , hence  $F_{r,r}$  is a normal subgroup of the centralizer  $C_W(r)$  as well.

Since any group having subgroups satisfying the hypotheses of Claims 1-3 decomposes as a direct product of the two subgroups, the result follows.  $\square$

**Lemma 6.3.5** *Let  $r, s, t \in S$ . If  $u \in F_{r,s}$  and  $v \in F_{s,t}$ , then  $vu \in F_{r,t}$ .*

Proof: Observe that  $vu\Sigma^r = \Sigma^t$  and that the chambers  $K$  and  $uK$  are on the same side of the wall  $\Sigma^s$ , while the chambers  $K$  and  $vK$  are on the same side of the wall  $\Sigma^t$  (by the same argument as in the proof of Lemma 6.3.2). Since  $v$  acts by isometry on the Davis complex  $\Sigma$ , the chambers  $vK$  and  $vuK$  are on the same side of  $\Sigma^t$ . Thus  $K$  and  $vuK$  are on the same side of  $\Sigma^t$ . By Lemma 3.1.8,  $l(tvu) > l(vu)$ , which implies that  $vu \in F_{r,t}$ .  $\square$

**Theorem 6.3.6** *Let  $(W, S)$  be a Coxeter system of type  $K_n$  and let  $a \in S$ . Given vertices  $uW_A, vW_A \in \Sigma^a$ , there exists an  $\alpha \in F_{a,a}$  such that  $\alpha uW_A = vW_A$ .*

Proof: We consider three cases.

Case 1: We first suppose that  $a \in A$  and that  $uW_A = W_A$ . In this case  $W_{\{a\}}$  is a subgroup of  $W_A$ . As in Lemma 6.2.1, let  $\bar{v}$  be the unique element of minimal length in  $vW_{\{a\}}$ . In this case

$$\bar{v}W_{\{a\}}a = vW_{\{a\}} \in \Sigma^a$$

(i.e.,  $\bar{v}W_{\{a\}}$  is a vertex in the wall  $\Sigma^a$ ), so  $a = (\bar{v})^{-1}\bar{v}$ . This implies that

$$(\bar{v})^{-1}\Sigma^a = \Sigma^{(\bar{v})^{-1}a\bar{v}} = \Sigma^a.$$

Applying Lemma 3.2.1 a, we obtain:

$$l(a(\bar{v})^{-1}) = l(\bar{v}a) > l(\bar{v}) = l((\bar{v})^{-1})$$

(the inequality is due to the minimality of  $\bar{v}$  in  $vW_{\{a\}}$ ). We may now conclude that  $(\bar{v})^{-1} \in F_{a,a}$ . Since  $(\bar{v})^{-1}vW_{\{a\}} = W_{\{a\}}$ , it follows that  $(\bar{v})^{-1}v \in W_A$  and, by letting  $\alpha = (\bar{v})^{-1}$  we see that

$$\alpha vW_A = W_A = uW_A$$

as desired.

Case 2: We continue to assume that  $a \in A$ . If  $uW_A \neq W_A$ , let  $\alpha, \beta \in F_{a,a}$  be such that  $\alpha uW_A = W_A$  and  $\beta vW_A = W_A$  (Case 1 provides the existence of the elements  $\alpha$  and  $\beta$ ). It follows that

$$\beta^{-1}\alpha uW_A = \beta^{-1}W_A = vW_A.$$

By Theorem 6.3.3,  $\beta^{-1}\alpha \in F_{a,a}$ . Which completes the proof in the case that  $a \in A$ .

Case 3: If  $a \notin A$ , pick  $t \in A$  and let

$$z = (at)^{\frac{m(a,t)-1}{2}}.$$

Then  $zaz^{-1} = t$ , and  $tz$  is the unique element in  $W_{\{a,t\}}$  of maximal length (by Lemma 6.2.4). Observe that  $z\Sigma^a = \Sigma^t$  and, by maximality,  $l(tz) > l(z)$ . Consequently,  $z \in F_{a,t}$ . Since  $uW_A$  and  $vW_A$  are vertices of  $\Sigma^a$ , it follows that  $zuW_A$  and  $zvW_A$  are vertices of  $\Sigma^t$ . Since  $t \in A$ , Case 2 applies: we find  $\beta \in F_{t,t}$  with  $\beta zuW_A = zvW_A$ . This means that  $z^{-1}\beta zuW_A = vW_A$  and, by Lemma 6.3.5,  $z^{-1}\beta z \in F_{a,a}$ .

This completes the proof. □

Recall that when  $(W, S)$  is a Coxeter system such that  $W$  is of type  $K_n$ , the nerve  $N(S)$  is isomorphic to the Coxeter graph  $\Gamma_S$ , which is the complete graph

on  $n$  vertices. Let  $sdN(S)$  be the graph obtained from  $N(S)$  by barycentrically subdividing  $N(S)$  and labelling by  $\{s, t\}$  the new vertex obtained in the subdivision of the edge joining  $s$  and  $t$  in  $N(S)$ . Note that  $sdN(S)$  has the homotopy type of  $N(S)$  (and of  $\Gamma_S$ ).

If  $a \in S$ , then  $C_W(a)$  acts on  $\Sigma^a$ , by Lemma 6.3.1. Restricting to our attention to the subgroup  $F_{a,a}$  of the centralizer  $C_W(a)$ , one notes that  $F_{a,a}$  acts on  $\Sigma^a$  as well. We now obtain a characterization of the quotient graph  $F_{a,a} \backslash \Sigma^a$ .

**Theorem 6.3.7** *The quotient graph  $F_{a,a} \backslash \Sigma^a$  is isomorphic to  $sdN(S)$ .*

Proof: Let  $V(\Sigma^a)$  and  $V(sdN(S))$  be the vertex sets of  $\Sigma^a$  and  $sdN(S)$ , respectively. Define

$$\bar{f} : V(\Sigma^a) \longrightarrow V(sdN(S))$$

by  $\bar{f}(wW_T) = T$  for all  $wW_T \in V(\Sigma^a)$  ( $\bar{f}$  is well-defined by Corollary 3.2.4).

This induces a simplicial map

$$f : \Sigma^a \longrightarrow sdN(S)$$

defined by the following.

1. If  $v \in V(\Sigma^a)$  is a vertex of  $\Sigma^a$ , then  $f(v) = \bar{f}(v)$ .
2. If  $e = \{wW_A, vW_B\}$  is an edge in  $\Sigma^a$ , then  $f(e)$  is the edge

$$f(e) = \{\bar{f}(wW_A), \bar{f}(vW_B)\} = \{A, B\}.$$

Let  $p : \Sigma^a \longrightarrow F_{a,a} \backslash \Sigma^a$  denote the projection map. We show that  $f$  and  $p$  respect each other's identifications. In this case a simplicial isomorphism

$$h : F_{a,a} \backslash \Sigma^a \longrightarrow sdN(S)$$

is induced.

First suppose that  $f(uW_A) = f(vW_B)$ . This implies that  $A = B$ . By Theorem 6.3.6, there is an  $\alpha \in F_{a,a}$  such that  $\alpha wW_A = vW_B$ . In other words,  $p(uW_A) = p(vW_B)$ .

Now assume that  $p(uW_A) = p(vW_B)$ . This means that there is an  $\alpha \in F_{a,a}$  such that  $\alpha uW_A = vW_B$ . Elementary group theory tells us that this is only possible if and only if  $W_A = W_B$ . By Corollary 3.2.4,  $A = B$ . Therefore  $f(uW_A) = f(vW_B)$ , and the result follows.  $\square$

Since  $\Sigma^a$  is simply connected, it is the universal cover of  $F_{a,a} \backslash \Sigma^a$ , and the subgroup  $F_{a,a}$  is the group of deck transformations corresponding to this cover. Standard covering space theory implies that  $F_{a,a} \cong \pi_1(F_{a,a} \backslash \Sigma^a)$  and, since the map

$$h : F_{a,a} \backslash \Sigma^a \longrightarrow sdN(S)$$

of Theorem 6.3.7 is an isomorphism,

$$\pi_1(sdN(S)) \cong \pi_1(F_{a,a} \backslash \Sigma^a) \cong F_{a,a}.$$

With this in mind we have the following result.

**Corollary 6.3.8** *If  $(W, S)$  is a Coxeter group of type  $K_n$  and  $a \in W$  is a reflection, then the subgroup  $F_{a,a}$  is a free group of rank  $\frac{n^2 - 3n + 2}{2}$ .*

Proof: The fundamental group of any graph is a free group with basis in one-to-one correspondence with the edges outside a maximal tree. Since  $sdN(S)$

has the homotopy type of  $N(S)$ , we may consider  $\pi_1(N(S))$ . By Lemma 5.1.1, the nerve  $N(S)$  is the complete graph on  $n$  vertices. It is easy to show that a complete graph on  $n$  vertices has  $\frac{n(n-1)}{2}$  edges and that the  $n-1$  vertices emanating from a single vertex form a maximal tree. Therefore the rank of  $\pi_1(N(S))$  is equal to

$$\frac{n(n-1)}{2} - (n-1) = \frac{n^2 - 3n + 2}{2},$$

and, since  $\pi_1(N(S)) \cong F_{a,a}$ , the result follows.  $\square$

## 6.4 A Reduction Process

Given a Coxeter system  $(W, S, m)$  of type  $K_n$  and  $a \in S$ , we are now in a position to characterize the collection of walls which meet the wall  $\Sigma^a$ . The key element in this characterization is Theorem 6.3.6. Assume

$$S = \{s_1, s_2, \dots, s_n\},$$

and  $a = s_1$ . The collection of maximal special subgroups of  $W$  is indexed by the set

$$\mathcal{Z} = \{\{s_i, s_j\} \in \mathcal{S}(S) : 1 \leq i < j \leq n\}.$$

If  $A = \{s_i, s_j\} \in \mathcal{Z}$ , then the special subgroup  $W_A$  is a dihedral group of order  $2m(s_i, s_j)$  and, for any  $w \in W$ , the coset  $wW_A$  is a vertex of the wall  $\Sigma^a$  whenever  $a \in wW_Aw^{-1}$ . Call such a vertex a *maximal vertex* of  $\Sigma^a$ . Let  $R_A$  be the set of reflections in  $W_A$  (recall that  $R_A = \{ubu^{-1} : u \in W_A, b \in A\}$ ).

For each  $A \in \mathcal{Z}$ , consider the set

$$\mathcal{Y}_A = \{uW_A \in \Sigma : a \in uW_Au^{-1}\}.$$

The set  $\mathcal{Y}_A$  is the collection of maximal vertices of  $\Sigma^a$  which are left cosets of  $W_A$  in  $W$ . By Theorem 6.3.7, vertices  $uW_A, vW_B \in \Sigma^a$  are equivalent under the action of  $F_{a,a}$  if and only if  $A = B$ . Thus, for each  $A \in \mathcal{Z}$ , and any coset  $uW_A \in \mathcal{Y}_A$ , the equivalence class  $[uW_A]$  of  $uW_A$  under the action of  $F_{a,a}$  on the wall  $\Sigma^a$  is equal to  $\mathcal{Y}_A$ .

**Theorem 6.4.1** *Let  $a \in S$  and  $p \in W$ . If  $\Sigma^p$  is a wall in the Davis complex  $\Sigma$  and  $\Sigma^p \cap \Sigma^a \neq \emptyset$ , then there exists  $A \in \mathcal{Z}$ , a reflection  $r \in R_A$ , an element  $\alpha \in F_{a,a}$ , and a coset  $u_A W_A \in \mathcal{Y}_A$  such that  $\Sigma^p = \alpha u_A \Sigma^r$ .*

In other words, the wall  $\Sigma^p$  can be expressed as a  $F_{a,a}$ -translate of another wall  $\Sigma^r$ , and the element  $r$  is reflection in the dihedral subgroup  $W_A$ .

Proof: For each  $A \in \mathcal{Z}$ , fix a representative  $u_A W_A \in \mathcal{Y}_A$  for the equivalence class  $[u_A W_A]$  under the  $(F_{a,a})$ -action on  $\Sigma^a$ . Let  $v$  be the vertex in which  $\Sigma^p$  meets  $\Sigma^a$  (that  $v$  is unique follows from the proof of Theorem 5.2.6). Since  $v \in \mathcal{Y}_A$  for some set  $A \in \mathcal{Z}$ , it follows that  $v = \alpha u_A W_A$  for some  $\alpha \in F_{a,a}$  (by Theorem 6.3.6). The walls that meet the vertex  $W_A$  in  $\Sigma$  are precisely the walls of the form  $\Sigma^r$  where  $r \in R_A$ . Therefore the walls which meet the vertex  $v$  in  $\Sigma$  are precisely the walls of the form  $\alpha u_A W_A \Sigma^r$ .  $\square$

As a consequence of Theorem 6.4.1, we may locate a finite collection of walls  $\mathcal{W}$  in the Davis complex  $\Sigma$  such that each wall meeting the wall  $\Sigma^a$  is a  $(F_{a,a})$ -translate of a wall in  $\mathcal{W}$ .

## 6.5 Application of the Reduction Process

Suppose  $W$  is a Coxeter group of type  $K_n$  with Coxeter systems  $(W, S)$  and  $(W, S')$ . Though we know that  $\Gamma_S$  and  $\Gamma_{S'}$  are isomorphic, we would like

to show that  $\Gamma_S$  and  $\Gamma_{S'}$  are graph-isomorphic (recall that this would imply rigidity of  $W$ ). In order to do so we must determine  $m'(p, q)$  for each pair of distinct generators  $p, q \in S'$  (cf. Section 4.2).

Fix a generator  $a \in S$ . As in the proof of Theorem 6.4.1, for each  $A \in \mathcal{Z}$ , we choose a representative  $u_A W_A$  for the equivalence class of the maximal vertices under the action of the group  $F_{a,a}$  on the wall  $\Sigma^a$ . For each  $p \in S'$ , we view  $\Sigma^p$  as a wall in  $\Sigma(S)$  (by Theorem 5.4.2, each generator in  $S'$  acts by reflection on  $\Sigma(S)$ ). We conclude that  $p = xbx^{-1}$  for some  $b \in S$  and  $x \in W$  (every reflection on  $\Sigma(S)$  is conjugate to an element of  $S$ ). Since  $W$  is of type  $K_n$ , the generator  $b$  is conjugate in  $W$  to  $a$  and we may assume without loss of generality that  $p = xax^{-1}$ . As a consequence,  $\Sigma^p = x\Sigma^a$ . Since the Coxeter system  $(W, S')$  is also of type  $K_n$ , given  $p, q \in S'$ , we know that  $W_{\{p,q\}}$  is a finite dihedral group. Corollary 4.1.3 implies that

$$\Sigma^{W_{\{p,q\}}} = \Sigma^p \cap \Sigma^q \neq \emptyset.$$

Since  $\Sigma^p = x\Sigma^a$ , it follows that

$$x\Sigma^a \cap \Sigma^q \neq \emptyset$$

and therefore

$$x^{-1}\Sigma^q \cap \Sigma^a \neq \emptyset.$$

Applying Theorem 6.4.1, for each  $q \in S' - \{p\}$  we write

$$\Sigma^q = \alpha_q u_{A_q} W_{A_q} \Sigma^{r_q},$$

for some  $\alpha_q \in F_{a,a}$ ,  $A_q \in \mathcal{Z}$ ,  $r_q \in R_{A_q}$ , and  $u_{A_q} W_{A_q}$  a representative of the equivalence  $[u_{A_q} W_{A_q}]$  under the action of  $F_{a,a}$  on  $\Sigma^a$ . In order to determine  $m'(p, q)$  it is necessary and sufficient to determine the order of the dihedral group  $W_{A_q}$ .

In approaching this problem, we are lead to consider the following question. Given  $A, B \in \mathcal{Z}$ , elements  $\alpha, \beta \in F_{a,a}$ , reflections  $r_A \in R_A$ ,  $r_B \in R_B$ , and  $u_A W_A, u_B W_B$  representatives of maximal vertices in  $\Sigma^a$ , when is

$$\alpha u_A \Sigma^{r_A} \cap \beta u_B \Sigma^{r_B}$$

nonempty? By applying  $(\alpha u_A)^{-1}$ , this is equivalent to deciding if

$$\Sigma^{r_A} \cap [(u_A)^{-1} \alpha^{-1} \beta u_B] \Sigma^{r_B}$$

is nonempty. By noting that  $t(r_A)t^{-1} = a$  for some  $t \in W$  (every reflection is conjugate to  $a$  and, when  $A$  is given explicitly, we can determine a finite list of choices for  $t$ ), we may further reduce the question to the following: when is

$$\Sigma^a \cap t^{-1}(u_A)^{-1} \alpha^{-1} \beta u_B \Sigma^{r_B}$$

nonempty?

By Theorem 6.4.1, if

$$\Sigma^a \cap t^{-1}(u_A)^{-1} \alpha^{-1} \beta u_B \Sigma^{r_B} \neq \emptyset$$

then there exists  $\gamma \in F_{a,a}$  and maximal vertex representative  $zW_C \in \Sigma^a$  such that

$$\gamma z \Sigma^y = t^{-1}(u_A)^{-1} \alpha^{-1} \beta u_B \Sigma^{r_B}$$

for some  $y \in R_C$ . Though this gives an explicit method for approaching the general rigidity problem for Coxeter groups of type  $K_n$ , it appears to be quite difficult to determine the existence of the triple  $(\gamma, zW_C, y)$  directly.

## 6.6 Conclusion

As mentioned in Section 5.4, Corollary 5.4.8 yields an infinite subclass of type  $K_n$  Coxeter groups which are rigid. In Chapter 6 we have discussed a

strategy for extending rigidity to the entire class of type  $K_n$  Coxeter groups. The characterization of the interaction of walls given by Theorem 6.4.1 provides a first step in understanding the complicated topology of the Davis complex corresponding to a type  $K_n$  Coxeter group. It is hoped that this analysis will facilitate the application of geometric methods (especially the theory of  $CAT(0)$  groups) in the future.

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