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# Centralizers of Coxeter Elements and Inner Automorphisms of Right-Angled Coxeter Groups

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**Abstract**

Let $W$ be a right-angled Coxeter group. We characterize the centralizer of the Coxeter element of a finite special subgroup of $W$. As an application, we give a solution to the generalized word problem for $Inn(W)$ in $Aut(W)$.

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## 1 Introduction

Let $(W, S)$ be a right-angled Coxeter system. If the special subgroup $W_T$ generated by $T \subseteq S$ is finite, then $W_T$ is an elementary abelian 2-group of rank $|T|$. The product of the elements of $T$, denoted $u_T$, is called the *Coxeter element* of $W_T$. In this article we describe the centralizer $C(u_T)$ of $u_T$. Specifically, we prove:

**Theorem**: *If $T \subseteq S$ is such that $W_T$ is finite, then $C(u_T) = W_T$.*

Though Coxeter groups have been studied extensively, their automorphism groups have received relatively little attention. We show how the result above can be used in the study of the automorphisms of right-angled Coxeter groups.

Some other results concerning the automorphism group of a right-angled Coxeter group appear in [10], [13] and [16].

If $W$ is a right-angled Coxeter group, then the group $Aut(W)$ of automorphisms of $W$ acts on the set of conjugacy classes of involutions in $W$. Following Tits [16], the kernel of this action is denoted by $Aut^\circ(W)$. Since $W$ is a CAT(0) group [12], the index of $Aut^\circ(W)$ in $Aut(W)$ is finite and there is a series

$$1 \trianglelefteq Inn(W) \trianglelefteq Aut^\circ(W) \trianglelefteq Aut(W)$$

of normal subgroups of $Aut(W)$. A presentation for $Aut^\circ(W)$ was given by Mühlherr in [13]. Using a generating set $\mathcal{A}$ obtained as a slight modification of Mühlherr's and the characterization of centralizers of Coxeter elements above, we construct an effective algorithm for deciding whether a word in the free group $F(\mathcal{A})$ represents an inner automorphism of $W$ under the natural mapping $F(\mathcal{A}) \longrightarrow Aut^\circ(W)$. In other words, we prove the following.

**Theorem** *The generalized word problem for $Inn(W)$ in $Aut^\circ(W)$ is solvable.*

Since the quotient $Aut(W)/Aut^\circ(W)$ is finite, techniques developed by Farb [9] imply the following.

**Corollary** *The generalized word problem for $Inn(W)$ in $Aut(W)$ is solvable.*

**Remark**: There are other ways to solve the generalized word problem for $Inn(W)$ in $Aut^\circ(W)$. For example, one can use the "distance matrix," defined in [13], to detect inner automorphisms of $W$. The algorithm given in this paper has the advantage of being both practical and elementary in nature. Indeed, as it relies only on the combinatorics of the Coxeter diagram, one can easily implement it "by hand."

Section 2 contains a brief review of Coxeter systems and groups. In Section 3 we develop the main combinatorial tool, which is a characterization of the centralizer of the "Coxeter element" of a maximal finite special subgroup. The algorithm for detecting inner autormorphims is developed in Section 4.

## 2 Coxeter systems and Coxeter groups

A *Coxeter system* is a triple $(W, S, m)$ where $W$ is a group, $S$ is a finite subset of $W$, and

$$m : S \times S \longrightarrow \{1, 2 \ldots, \infty\}$$

such that

1. $m(s, t) = 1$ if and only if $s = t$;

2. $m(s,t) = m(t,s)$ for all $s, t \in S$;

3. $W$ has a presentation of the form $\langle S \mid (st)^{m(s,t)} \text{ for all } s,t \in S\rangle$ (there is no relator when $m(s,t) = \infty$).

The group $W$ is called a *Coxeter group.* When there is no confusion, we omit reference to "$m$" and call $(W,S)$ a Coxeter system. Given a Coxeter system $(W,S,m)$, let $\Gamma_S$ be the graph defined as follows:

1. The vertex set of $\Gamma$ is $S$.

2. Distinct vertices $s,t \in S$ are joined by the (undirected) edge $\{s,t\}$ if and only if $m(s,t) < \infty$.

3. The edge $\{s,t\}$ is labeled $m(s,t)$ if and only if $m(s,t) > 2$.

The graph $\Gamma_S$, called the *Coxeter diagram* of $(W,S,m)$, provides a convenient method of encoding a Coxeter system diagrammatically. A Coxeter system $(W,S,m)$ such that $m(s,t) = 2$ or $m(s,t) = \infty$ whenever $s,t \in S$ are distinct is said to be a *right-angled Coxeter system*. The diagram of a right-angled Coxeter system is therefore an unlabeled graph. We remark that there is a standard metric $d_\Gamma$ on the diagram of a right-angled Coxeter group that assigns length one to each edge. That there is a one-to-one correspondence between right-angled Coxeter groups and their diagrams is not obvious. A result due to D. Radcliffe [14] shows that this is, in fact, the case. Thus, a Coxeter group is said to be *right-angled* if it has a right-angled Coxeter system.

Since $s = s^{-1}$ in $W$ for all $s \in S$, each element $w \in W$ has an expression as a positive word of the form $w = s_1 s_2 \ldots s_n$ where $s_i \in S$ for all $1 \leq i \leq n$. We say that such an expression for $w$ is *reduced* if $n$ is as small as possible; the number of symbols in a reduced expression for $w$ is called the *length* of $w$, denoted $\ell(w)$.

The following result is fundamental to the theory of Coxeter groups. Proofs may be found in [3], [5] or [11]

**Theorem 2.1 (The Deletion Condition)** *Let $(W,S)$ be a Coxeter system and suppose that $w = s_1 \cdots s_n$ is a word in the generating set $S$. If $w$ is not reduced then there exist indices $1 \leq i < j \leq n$ such that the words*

$$w' = s_1 s_2 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_n$$

*and $w = s_1 \cdots s_n$ represent the same element of $W$.*

**Remark 2.2** It is clear that, in general, the deleted symbols $s_i$ and $s_j$ above are conjugate in $W$. However, if $(W,S)$ is right-angled and $s \in S$, there is a retraction $W \longrightarrow W_{\{s\}}$ whose kernel is the normal closure of $S - \{s\}$ (cf. [2]). It follows that elements of $S$ are conjugate in $W$ if and only if they are equal.

**Remark 2.3** In [17], Tits showed that a consequence of the Deletion Condition is the solvablity of the word problem for Coxeter groups.

Let $(W, S)$ be a Coxeter system with diagram $\Gamma_S$. If $T \subseteq S$, then $W_T$ denotes the subgroup of $W$ generated by $T$ (by definition, the subgroup generated by the empty set is the trivial subgroup); such subgroups are called *special subgroups*. The significance of the special subgroups is demonstrated by the following results (for proofs, see [5] or [11]).

**Theorem 2.4** *Let $A, B \subseteq S$. Then*

(a) $W_A \cap W_B = W_{A \cap B}$.

(b) *If $m_A$ is the restriction of $m$ to $A \times A$, then $W_A$ is a Coxeter group with Coxeter system $(W_A, A, m_A)$. Moreover, the subgraph of $\Gamma_S$ spanned by $A$ is a Coxeter diagram for $W_A$.*

(c) *If $(W, S)$ is right-angled, then the following are equivalent:*

  (i) *$W_A$ is finite.*

  (ii) *The subgraph of $\Gamma_S$ spanned by $A$ is complete.*

  (iii) *$W_A$ is an elementary abelian 2-group of rank $|A|$.*

# 3 Right-angled Coxeter groups

For the remainder of this article $(W, S)$ denotes a right-angled Coxeter system with diagram $\Gamma_S$. We consider the centralizers of certain elements of $W$; it turns out that these centralizers are conveniently described in terms of the Coxeter diagram. If $w \in W$, then $C(w)$ denotes the centralizer of $w$ in $W$.

For each $s \in S$, we define the following subsets of $S$:

$$s^* = \{t \in S \mid d_\Gamma(s,t) \leq 1\}$$

and

$$s^\perp = \{t \in S \mid d_\Gamma(s,t) > 1\}.$$

The next result follows from the work of Brink [6].

**Theorem 3.1 (B. Brink [6])** *If $(W, S)$ is a right-angled Coxeter system, then $C(s) = W_{s^*}$ for each $s \in S$.*

**Lemma 3.2** *Let $t \in S$ and $w \in W$ such that $tw = wt$. If $w = s_1 \cdots s_n$ is a reduced expression for $w$, then $ts_i = s_it$ for each $1 \leq i \leq n$.*

Proof: Since $w \in C(t)$, we have that $w \in W_{t^*}$ by Theorem 3.1. The result now follows immediately from [11], Theorem 5.5 (b). □

As noted above, special subgroups and their corresponding subgraphs play a key role. Of particular interest is the set

$$\mathcal{M}(S) = \{T \subseteq S \mid W_T \text{ is a maximal finite subgroup}\}.$$

A consequence of Theorem 2.4 (c) is that the maximal finite special subgroups of $W$ are in one-to-one correspondence with the maximal complete subgraphs of $\Gamma_S$. For each $T \in \mathcal{M}(S)$, let

$$u_T = \prod_{t \in T} t$$

Since $W_T$ is abelian, $u_T$ is independent of the ordering of $T$. We call $u_T$ the *Coxeter element* of $W_T$.

**Lemma 3.3** *If $T \in \mathcal{M}(S)$, then $u_T$ is an involution in $W$ and $\ell(u_T) = |T|$.*

Proof: Since $W_T$ is abelian, it follows that $(u_T)^2 = 1$ in $W$. The second assertion follows immediately from the Deletion Condition and Remark 2.2. □

We are now able to characterize the centralizers of Coxeter elements.

**Theorem 3.4** *Let $T \in \mathcal{M}(S)$ and $x \in C(u_T)$. If $x = s_1 \cdots s_n$ is a reduced expression for $x$ then, $s_i t = t s_i$ for all $1 \leq i \leq n$, and all $t \in T$.*

Proof: The proof relies on elementary properties of the length function (see e. g., [11]). If $T = \{t_1, \ldots, t_m\}$ we write $u_T = t_1 \cdots t_m$ and proceed by induction on $\ell(x)$. If $\ell(x) = 1$, then we have

$$s_1(t_1 \cdots t_m) = (t_1 \cdots t_m)s_1.$$

By Lemma 3.3, $u_T = t_1 \cdots t_m$ is a reduced expression for $u_T$ and so Lemma 3.2 applies.

Now assume the result holds for all elements of $C(u_T)$ of length less than $n$. There are two cases to consider.

Case 1: If $\ell(xt_i) < \ell(x)$ for some $1 \leq i \leq m$, then $\ell(xt_i) = n - 1$ and $xt_i = s_1 \cdots s_n t_i$ is not reduced. Hence, we may apply the Deletion Condition to obtained a shorter expression for $xt_i$. Since $s_1 \cdots s_n$ is reduced, one of the deleted letters must be $t_i$ and so $t_i = s_j$ for some $1 \leq j \leq n$. In particular, $s_j t = t s_j$ for all $t \in T$. Now we have

$$\begin{aligned} u_T &= x u_T x^{-1} \\ &= (s_1 \cdots s_n t_i) u_T (t_i s_n \cdots s_1) \\ &= (s_1 \cdots s_{j-1} s_{j+1} \cdots s_n) u_T (s_n \cdots s_{j+1} s_{j-1} \cdots s_1) \end{aligned}$$

As $(s_1 \cdots s_{j-1}s_{j+1} \cdots s_n)$ is reduced, it follows by induction that $s_i t = t s_i$ for all $i \neq j$ and all $t \in T$.

Case 2: If $\ell(xt_i) > \ell(x)$ for all $1 \leq i \leq m$, then it follows that

$$xu_T = (s_1 \cdots s_n)(t_1 \cdots t_m)$$

is a reduced expression for $xu_T$ (see e. g., [8], Lemma 1.3). Since

$$(s_1 \cdots s_n)(t_1 \cdots t_m)s_n = (t_1 \cdots t_m)(s_1 \cdots s_{n-1}),$$

the expression on the left-hand side is not reduced. By the Deletion Condition, there is an integer $1 \leq j < n$ such that $s_j = s_n$. It follows that

$$\begin{aligned}(s_{j+1} \cdots s_n)(t_1 \cdots t_m) &= (s_j \cdots s_n)(t_1 \cdots t_m)s_n \\ &= s_n(s_{j+1} \cdots s_n)(t_1 \cdots t_m)s_n\end{aligned}$$

in $W$. Moreover, since $(s_1 \cdots s_n)(t_1 \cdots t_m)$ is reduced, so is $(s_{j+1} \cdots s_n)(t_1 \cdots t_m)$. Lemma 3.2 implies that $s_n t_i = t_i s_n$ for all $1 \leq i \leq m$.

The result now follows by induction. □

We conclude this section with characterizations of the centralizers of Coxeter elements and the center of $W$ in terms of $\mathcal{M}(S)$. The proof of each is a straightforward application of Theorem 3.4.

**Corollary 3.5** *If $T \in \mathcal{M}(S)$, then $C(u_T) = W_T$.*

**Corollary 3.6** *If $Z(W)$ denotes the center of $W$, then $Z(W) = \bigcap_{T \in \mathcal{M}(S)} W_T$.*

# 4 Detecting Inner Automorphisms

Let $G$ be a group with finite generating set $A$ and let $H$ be a finitely generated subgroup of $G$. If there is an algorithm that decides whether or not a word in the free group with basis $A$ represents an element of $H$, then the *generalized word problem for $H$ in $G$* is said to be *solvable*.

For any group $G$, let $Aut^\circ(G)$ denote the kernel of the obvious action of $Aut(G)$ on the set of conjugacy classes of involutions of $G$. Clearly, $Inn(G)$ is a normal subgroup of $Aut^\circ(G)$ and so there is a normal series

$$1 \trianglelefteq Inn(G) \trianglelefteq Aut^\circ(G) \trianglelefteq Aut(G).$$

In a Coxeter group, there are only finitely many conjugacy classes of involutions of (this follows from [15] or the fact that Coxeter groups are CAT(0) groups [12]). Therefore, if $G$ is any Coxeter group (not necessarily right-angled), then $Aut^\circ(G)$ has finite index in $Aut(G)$.

In [13], Mühlherr gives a presentation for $Aut^\circ(W)$ in the case that $W$ is a right-angled Coxeter group. We describe a set $\mathcal{A}$ of generators for $Aut^\circ(W)$ obtained as a slight modification of that given by Mühlherr. We then provide a solution to the generalized word problem for $Inn(W)$ in $Aut(W)$.

We assume that the diameter of $\Gamma_S$ is greater than one (otherwise, $Inn(W)$ is trivial). For each $s \in S$, the subgraph of $\Gamma_S$ spanned by $s^\perp$ consists of a nonempty collection of components. If $K$ is the vertex set of one such component, then the mapping $\overline{\sigma_{sK}} : S \longrightarrow W$ defined by

$$\overline{\sigma_{sK}}(t) = \begin{cases} sts & \text{if} \quad t \in K \\ t & \text{if} \quad t \notin K \end{cases} \tag{1}$$

extends uniquely to an involutory automorphism $\sigma_{sK}$ of $W$. It follows from [13] that the set

$$\mathcal{A} = \{\sigma_{sK} \mid s \in S,\ K \text{ a component of } s^\perp\}$$

generates $Aut^\circ(W)$.

For each $\tau \in Inn(W)$, fix an element $a_\tau \in W$ such that $\tau(x) = a_\tau x a_\tau^{-1}$ for all $x \in W$. Note that, if $v \in W$ is an involution, then $\tau(v)$ is conjugate to $v$ in $S$. An *initial segment* for $\tau(v)$ is a reduced expression $\omega_v$ such that $\tau(v) = \omega_v v \omega_v^{-1}$ in $W$. A key observation is the following: $a_\tau \in \omega_v C(v)$ for every involution $v \in W$. In particular,

$$a_\tau \in \bigcap_{T \in \mathcal{M}(S)} \omega_{u_T} C(u_T). \tag{2}$$

**Theorem 4.1** *The generalized word problem for $Inn(W)$ in $Aut^\circ(W)$ is solvable.*

Proof: Let $\tau \in Aut^\circ(W)$. Since the elements of $\mathcal{A}$ are involutions of $Aut^\circ(W)$, there exist $\sigma_{s_1K_1}, \ldots, \sigma_{s_nK_n} \in \mathcal{A}$ such that $\tau = \sigma_{s_1K_1} \cdots \sigma_{s_nK_n}$ in $Aut^\circ(W)$ (in other words, every element of $Aut^\circ(W)$ can be expressed as a positive word in the free group with basis $\mathcal{A}$). Using (1) and the Deletion Condition, the initial segment $\omega_{u_T}$ of the Coxeter element $u_T$ can be computed for each $T \in \mathcal{M}(S)$. By Corollary 3.5, $\omega_{u_T} C(u_T) = \omega_{u_T} W_T$ and so the intersection

$$\bigcap_{T \in \mathcal{M}(S)} \omega_{u_T} C(u_T) \tag{3}$$

is a finite set of elements in $W$. Using the Deletion Condition, one can enumerate the elements of (3) as a list of positive reduced expressions. If this intersection is empty, then it follows from (2) that $\tau \notin Inn(W)$. Otherwise, we write

$$\bigcap_{T\in\mathcal{M}(S)} \omega_{u_T} C(u_T) = \{z_1, \ldots z_r\}$$

where each $z_i$ is given as a positive reduced expression. Again, applying the Deletion Condition, the question of whether there exists an integer $1 \leq i \leq r$ such that

$$\omega_s s \omega_s^{-1} = z_i s z_i^{-1}$$

in $W$ for all $s \in S$ is decidable. □

It is well-known that $W$ has a faithful representation in $GL(V)$, where $V$ is a finite-dimensional real vector space (cf. [5]). It follows that $W$ is residually finite (see e. g., [1]). Since the automorphism group of a finitely generated residually finite group is residually finite [4], it follows that $Aut(W)$ has solvable word problem.

**Corollary 4.2** *The generalized word problem for $Inn(W)$ in $Aut(W)$ is solvable.*

Proof: Since $Aut^\circ(W)$ has finite index in $Aut(W)$, it follows that $Aut^\circ(W)$ is a quasiconvex subgroup of $Aut(W)$ (with respect to any finite set of generators for $Aut(W)$). The result now follows immediately from Theorem 4.1 above, and Propositions 2.1 and 4.2 of [9]. □

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