A Class of Rigid Coxeter Groups

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Abstract

A Coxeter group W is said to be *rigid* if, given any two Coxeter systems (W, S) and (W, S'), there is an automorphism $\rho : W \longrightarrow W$ such that $\rho(S) = S'$. We consider the class of Coxeter systems (W, S) for which the Coxeter graph Γ_S is complete and has only odd edge labels (such a system is said to be of "type K_n"). It is shown that if W has a type K_n system, then any other system for W is also type K_n. Moreover, the multi-set of edge labels on Γ_S and $\Gamma_{S'}$ agree. In particular, if all but one of the edge labels of Γ_S are identical, then W is rigid.

1 Coxeter Group Preliminaries

1.1 Coxeter Systems

A *Coxeter system* is a triple (W, S, m) where W is a group, S is a subset of W, and

$$m : S \times S \longrightarrow \{1, 2, 3, \ldots, \infty\}$$

is a function satisfying

(a) $m(s,t) = 1$ if and only if $s = t$;
(b) $m(s,t) = m(t,s)$ for all $(s,t) \in S \times S$;
(c) W has a presentation of the form

$$< S : (st)^{m(s,t)} = 1\,; (s,t) \in S \times S >$$

(if $m(s,t) = \infty$ for some $s, t \in S$, then the elements st and ts have infinite order in W, and there is no corresponding relation included in the presentation). The group W is called a *Coxeter group*. When there is no confusion, the "m" will often be omitted from the notation and the pair (W, S) will be referred to as a Coxeter system. We assume throughout that S is a finite set. Let $R_S = \{wsw^{-1} : w \in W,\ s \in S\}$. If $r \in R_S$, then r is called a *reflection* (with respect to the generating set S).

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There are a number of conventions for encoding the data given in a Coxeter system diagramatically. The following will be used throughout: given a Coxeter system (W, S), the *Coxeter graph* is the labeled graph Γ_S which satisfies

1. the vertex set of Γ_S is (in one-to-one correspondence with) S;
2. vertices $s, t \in S$ are joined by the edge $\{s, t\}$ if and only if $m(s, t) \geq 3$;
3. the edge $\{s, t\}$ is labeled by $m(s, t)$ if and only if $m(s, t) \geq 4$.

We shall not distinguish between the subset $\{s, t\} \subseteq S$ and the edge $\{s, t\}$ of Γ_S.

A Coxeter system (W, S) is *irreducible* if Γ_S is connected. Irreducible systems for which W is finite have been completely classified in terms of their graphs (see eg. [12, Chapter 2]). The next result is a direct consequence of [12, Theorem 6.4].

Lemma 1.1.1 *If (W, S) is irreducible and W is finite, then Γ_S is a tree.*

If (W, S, m) and (W', S', m') are Coxeter systems and $\theta : S \longrightarrow S'$ is a bijection, consider the following conditions:

C1. $\{s, t\}$ is an edge in Γ_S if and only if $\{\theta(s), \theta(t)\}$ is an edge in $\Gamma_{S'}$;
C2. $m(s, t) = m'(\theta(s), \theta(t))$.

If θ satisfies C1, then Γ_S and $\Gamma_{S'}$ are said to be *weakly isomorphic*. If θ satisfies both C1 and C2, then Γ_S and $\Gamma_{S'}$ are *isomorphic*.

1.2 The Rigidity Question

A Coxeter group W is *rigid* if, given any two systems (W, S) and (W, S') for W, there is an automorphism $\rho : W \longrightarrow W$ such that $\rho(S) = S'$. Clearly W is rigid if and only if the graphs Γ_S and $\Gamma_{S'}$ are isomorphic. Not all Coxeter groups are rigid: one can easily show that both Coxeter graphs in Figure 1 determine the dihedral group D_6 (the symmetry group of a regular hexagon).

Figure 1: Each graph determines D_6.

In fact, one can easily show that for any $k \geq 1$, the dihedral group D_{4k+2} is not rigid. Recently, B. Mühlherr [14] showed that the groups determined by the graphs in Figure 2 on page 3 are isomorphic.

Figure 2: Mühlherr's Example

In [3] N. Brady, J. McCammond, B. Mühlherr, and W. Neumann have shown that the example of Figure 2 actually belongs to a larger class of non-rigid Coxeter groups, namely those which may be obtained by "diagram twisting." This result significantly expands the list of known non-rigid Coxeter groups.

In the positive direction there have also been some recent developments.

Theorem 1.2.1 (D. Radcliffe, [17]) *If (W, S, m) is a Coxeter system such that $m(s, t) \in \{2, \infty\}$ for all $s, t \in S$, then W is rigid.*

A system satisfying the hypothesis of Theorem 1.2.1 is called *right-angled*. Radcliffe has extended his result to allow $m(s, t)$ to be a multiple of 4 as well.

Theorem 1.2.2 (R. Charney and M. Davis, [7]) *Suppose that W capable of acting effectively, properly, and cocompactly on a contractible mainifold. If (W, S) and (W, S') are any Coxeter systems for W, then there is a unique $w \in W$ such that $wSw^{-1} = S'$.*

Recall that a graph is *complete* if each pair of vertices is joined by an edge. Define a Coxeter system (W, S, m) to be of *type K_n* if

a) the Coxeter graph Γ_S is the complete graph on n vertices;
b) $m(s, t)$ is odd for all $s, t \in S$.

The main result of this paper is the following, proved as Theorem 3.4.7 below.

Main Theorem. *Let (W, S) be a Coxeter system of type K_n. If (W, S') is any other system for W, then it is type K_n (in particular the Coxeter graphs Γ_S and $\Gamma_{S'}$ are weakly isomorphic). Moreover, the multi-sets of edge labels on Γ_S and $\Gamma_{S'}$ are equal.*

A *multi-set* is a collection in which the orders of the elements do not matter but the multiplicities do. Thus the multi-sets $\{3, 5\}$, $\{3, 3, 5\}$ and $\{3, 5, 5\}$ are all distinct.

With the Main Theorem in hand one can easily see that if (W, S) is a type K_n system such that all but (at most) one edge label of Γ_S is identical, then W is rigid (Corollary 3.4.9). In particular, for any $k \geq 1$, the dihedral group D_{2k+1} is rigid (Corollary 3.4.8).

2 Geometry and Coxeter Groups

2.1 Reflection Groups

Let X be a connected, locally path connected, Hausdorff space. A *reflection* on X is an involution $r : X \longrightarrow X$ such that

a) the fixed-point set X^r of r separates X into two components which are interchanged by r;
b) every $x \in X^r$ has an arbitrarily small open neighborhood which is separated by X^r into two components which are interchanged by r.

The subspace X^r is called the *wall* (or *mirror*) corresponding to r.

A group Γ of homeomorphisms of X is a *reflection group* on X if

a) Γ is generated by reflections;
b) the collection of walls $\{X^r\}$ of all reflections $r \in \Gamma$ is locally finite family in X;
c) if r_1 and r_2 are distinct reflections in Γ, then any path α in X can be approximated arbitrarily closely by a path which misses the intersection $X^{r_1} \cap X^{r_2}$ (the intersection of two walls has "codimension 2").

It is well-known that any reflection group is a Coxeter group. More precisely, if Γ is a reflection group on a space X, there is a subset $V \subset \Gamma$ and a function

$$m : V \times V \longrightarrow \{1, 2, \dots, \infty\}$$

such that (Γ, V, m) is a Coxeter system (see [2] for the details). Conversely, given any Coxeter system (W, S), the group W acts as a reflection group on a convex open subset of a real vector space (the interior of the "Tit's cone," see [2] or [12]). Hence the terms reflection group and Coxeter group may be considered synonomous.

We will discuss the construction due to M. Davis [9] of a simplicial complex $\Sigma(S)$ on which W acts simplicially as a reflection group. This construction has numerous applications in topology and group theory. In particular, a result of G. Moussong (stated as Theorem 2.3.2 below) facilitates the use of geometric techniques to analyze Coxeter systems. For more details on reflections and Coxeter systems the reader is referred to [2], [6], or [12].

2.2 Special Subgroups

Let (W, S) be a Coxeter system. If $T \subseteq S$, write W_T for the subgroup of W generated by T. Define $W_\emptyset$ to be the trivial subgroup. If W_T is finite it is called a *special subgroup*.

The following results are particularly useful in the theory of Coxeter groups. Proofs may be found in [12, Chapter 5].

Theorem 2.2.1 *Let $A, B, T \subseteq S$, and let $m|_T$ be the restriction of m to the product $T \times T$. Then*

(a) W_T is a Coxeter group with Coxeter system $(W_T, T, m|_T)$;

(b) $W_A \cap W_B = W_{A \cap B}$.

As in [7], let $\mathcal{S}(S) = \{T \subseteq S : W_T \text{ is finite}\}$. In other words, $\mathcal{S}(S)$ is the collection of subsets of S which generate finite subgroups of W. If $w \in W$ and $T \in \mathcal{S}(S)$, the conjugate subgroup wW_Tw^{-1} is called an *S-parabolic subgroup* (or simply a *parabolic subgroup* when no confusion arises). Let

$$P(S) = \{wW_Tw^{-1} : w \in W,\ T \in \mathcal{S}(S)\}$$

be the collection of parabolic subgroups of W. If $G = wW_Tw^{-1}$, then the *parabolic rank* of G, denoted by $rk(G)$, is the cardinality of T. For $k \geq 0$, let

$$P_k(S) = \{G \in P(S) : rk(G) = k\}$$

be the set of parabolic subgroups in W of parabolic rank k.

2.3 Simplicial Complexes Associated with Coxeter Systems

Given a simplicial complex X with vertex set $V = V(X)$, let $C(X)$ be the collection of simplices of X together with the empty set. The set $C(X)$ is partially ordered by inclusion. Throughout we will identify a simplex with its vertex set: if $T \subset V$, then $T \in C(X)$ if and only if T spans a simplex in X or $T = \emptyset$.

If $\mathcal{P}$ is a poset and $\alpha \in \mathcal{P}$, define $\mathcal{P}_{\geq\alpha}$ to be the subposet of $\mathcal{P}$ of elements greater than or equal to α. Subposets $\mathcal{P}_{>\alpha}$, $\mathcal{P}_{\leq\alpha}$, and $\mathcal{P}_{<\alpha}$ are defined similarly. A poset $\mathcal{P}$ is an *abstract simplicial complex* if it is isomorphic to $C(X)_{>\emptyset}$ for some simplicial complex X. The complex X is the *realization* of $\mathcal{P}$.

Given a poset $\mathcal{P}$, the *derived complex* $\mathcal{P}'$ is the set of finite chains in $\mathcal{P}$. With a partial ordering of inclusion, $\mathcal{P}'$ is an abstract simplicial complex. The realization of $\mathcal{P}'$ is called the *geometric realization* of $\mathcal{P}$ and is denoted by $geom(\mathcal{P})$.

Fix a Coxeter system (W, S) and partially order $\mathcal{S}(S)$ by inclusion. The subposet $\mathcal{S}(S)_{>\emptyset}$ is an abstract simplicial complex. Let $N(S)$ be its realization. We will not distinguish between the subset $T \subseteq S$ and the simplex T of $N(S)$. Also, let

$$K = geom(\mathcal{S}(S)).$$

The complex $N(S)$ is called the *nerve* corresponding to the system (W, S). Note that K is the cone on the barycentric subdivision of $N(S)$. As in [17], the collection of maximal simplices in the nerve $N(S)$ is denoted by $N^*(S)$.

We are now in a position to describe the complex $\Sigma(S)$ mentioned above. Consider the poset

$$W\mathcal{S}(S) = \{wW_T : w \in W,\ T \in \mathcal{S}(S)\},$$

where the partial ordering is by inclusion. The simplicial complex

$$\Sigma = \Sigma(S) = geom(W\mathcal{S}(S))$$

is called the *Davis complex*. Thus there is a k-simplex in Σ for each chain in $W\mathcal{S}(S)$ of the form

$$w_0 W_{T_0} < w_1 W_{T_1} < \cdots < w_k W_{T_k}.$$

We shall not distinguish between the coset $wW_T \in W\mathcal{S}(S)$ and the vertex $wW_T \in \Sigma$. The group W acts naturally on Σ by left translation.

The correspondence $T \longmapsto W_T$ defines an embedding $\mathcal{S}(S) \longrightarrow W\mathcal{S}(S)$ which induces an embedding $K \longrightarrow \Sigma$. Thus K may be regarded as a subcomplex of Σ (K is identified with $geom(W\mathcal{S}(S)_{\geq W_\emptyset})$). Translates of K under the action of W are called *chambers* and K is called the *fundamental chamber*. Note that W acts freely on the set of chambers of Σ (i.e., for any chamber K_0 and $w \in W$, $wK_0 = K_0$ if and only if $w = 1$).

Theorem 2.3.1 (M. Davis, [9]) *Given a Coxeter system (W, S), the group W acts simplicially on Σ as a reflection group. The set of reflections on Σ is precisely R_S.*

As in Section 2.1, the wall corresponding to a reflection r is denoted by Σ^r. If G is a subgroup of W, let Σ^G be the subspace of Σ fixed point-wise by G. That is

$$\Sigma^G = \{x \in \Sigma : g \cdot x = x \text{ for all } g \in G\}.$$

The following result of G. Moussong enables the application of geometric techniques to the theory of Coxeter groups.

Theorem 2.3.2 (G. Moussong, [13]) *There is a piecewise Euclidean metric d_M on Σ such that (Σ, d_M) is a complete $CAT(0)$-space. Moreover, the action of W on Σ is proper, cocompact, and by isometries.*

We shall use the following two facts from the theory of $CAT(0)$-spaces. The reader is referred to [5] for the details. Recall that if X is a metric space and $x, y \in X$, a *geodesic arc* connecting x and y is an isometric embedding $\alpha : [a, b] \longrightarrow X$ of an interval $[a, b] \subseteq \mathbb{R}$ such that $\alpha(a) = x$ and $\alpha(b) = y$. The image of α is called a *geodesic segment* and is denoted by $[x, y]$.

Theorem 2.3.3 *Let X be a $CAT(0)$-space.*

(a) If $x, y \in X$, then there is a unique geodesic arc connecting x and y.

(b) If G is a finite group acting by isometries on X and X is complete, then the fixed-point set of G in X is nonempty.

Parabolic subgroups of Coxeter systems will play a large role in the discussion of the next section. We conclude this section with three useful results.

Lemma 2.3.4 *If H is any finite subgroup of W, then there is a parabolic subgroup G such that $\Sigma^H = \Sigma^G$.*

A proof of Lemma 2.3.4 may be found in [7, Prop. 1.1, (iii)].

Corollary 2.3.5 *The intersection of parabolic subgroups is a parabolic subgroup.*

Proof: If (W, S) is a Coxeter system and $G_1, G_2 \in P(S)$, then there exist $u, v \in W$ and $A, B \in \mathcal{S}(S)$ such that $G_1 = uW_A u^{-1}$ and $G_2 = vW_B v^{-1}$. As in Lemma 2.3.4, let $H \in P(S)$ be such that $\Sigma^H = \Sigma^{G_1 \cap G_2}$. The vertices uW_A and vW_B are fixed by H. As a consequence $H \subseteq G_1 \cap G_2$. But H is the maximal subgroup fixing Σ^H, implying that $G_1 \cap G_2 \subseteq H$. □

Lemma 2.3.6 *If $G_1, G_2 \in P(S)$ and G is the group generated by $G_1 \cup G_2$, then $\Sigma^{G_1} \cap \Sigma^{G_2} = \Sigma^G$.*

Proof: Clearly $\Sigma^G \subseteq \Sigma^{G_1} \cap \Sigma^{G_2}$. Choosing $x \in \Sigma^{G_1} \cap \Sigma^{G_2}$ and $g \in G$. Then g may be expressed as

$$g = g_1 g_2 \cdots g_k$$

for some positive integer k where each $g_i \in G_1 \cup G_2$. Thus

$$g \cdot x = (g_1 g_2 \cdots g_k) \cdot x = x,$$

whence $\Sigma^{G_1} \cap \Sigma^{G_2} \subseteq \Sigma^G$. □

3 Coxeter Graphs of Type K_n Coxeter Systems

This chapter will be devoted to showing that if (W, S) is a Coxeter system of type K_n and (W, S') is any other system for W, then Γ_S and $\Gamma_{S'}$ are weakly isomorphic. In certain cases we may conclude that W is rigid (Corollaries 3.4.8 and 3.4.9 below).

3.1 The Nerve of a Type K_n System

For the reader's convenience we repeat the following definition: a Coxeter system (W, S) is of *type* K_n if

a) Γ_S is the complete graph on $|S| = n$ vertices;
b) $m(s, t)$ is odd for all $s, t \in S$.

Lemma 3.1.1 *If (W, S) is a Coxeter system of type K_n, then the nerve $N(S)$ is the complete graph on n vertices.*

Proof: The vertex set of $N(S)$ is equal to S, so $N(S)$ has exactly n vertices. Given vertices s, t in $N(S)$, the subgroup $W_{\{s,t\}}$ is finite ($m(s, t)$ is odd), and we may conclude that s and t are joined by the edge $\{s, t\}$ in $N(S)$. If $a, b, c \in S$, then a, b, c are the vertices of a cycle in Γ_S (since Γ_S is complete). By Lemma 1.1.1, the subgroup $W_{\{a,b,c\}}$ is infinite. Hence any subset $T \subset S$ of cardinality greater than two generates an infinite subgroup of W. In other words, the set T does not span a simplex in $N(S)$. Therefore $N(S)$ is a graph without loops on n vertices having the property that every pair of vertices is joined by a unique edge, completing the proof. □

3.2 Conjugacy of Maximal Special Subgroups

Given a Coxeter system (W, S) and a finite subgroup G of W, let $F(G)$ be the subposet of $W\mathcal{S}(S)$ fixed by G. Specifically,

$$\begin{aligned} F(G) &= \{wW_T \in W\mathcal{S}(S) : G \cdot wW_T = wW_T\} \\ &= \{wW_T \in W\mathcal{S}(S) : G \subseteq wW_T w^{-1}\}. \end{aligned}$$

Lemma 3.2.1 *If G is a maximal finite subgroup of W, then $G \in P(S)$.*

Proof: Since G is finite and acts by isometries on Σ, the fixed-point set Σ^G is nonempty (Theorem 2.3.3 (b)). Since G acts freely on the set of chambers of Σ and this action is simplicial we deduce that G must fix a vertex v of Σ. Since every vertex v of Σ is of the form $v = wW_T$ for some $w \in W$ and $T \in \mathcal{S}(S)$ we have

$$G \cdot v = G \cdot wW_T = wW_T.$$

Consequently $G \subseteq wW_T W^{-1}$ and, since G is maximal, equality holds. □

The following result of Charney and Davis characterizes certain subposets of $W\mathcal{S}(S)$.

Theorem 3.2.2 (R. Charney and M. Davis, [7]) *Let $A \in \mathcal{S}(S)$ and $wW_T \in F(W_A)$. Then the subposet $F(W_A)_{\leq wW_T}$ is isomorphic to the poset of faces of a convex cell of dimension $|T| - |A|$.*

Inherent in the structure of Coxeter groups of type K_n is the fact that all maximal parabolic subgroups have equal parabolic rank. In geometric terms, this means that each maximal simplex in the nerve has the same dimension. It is also apparent that the nature of the collection of parabolic subroups is intrinsic to the topology of the Davis complex: each simplex in Σ corresponds to a chain of cosets

$$w_1 W_{T_1} < w_2 W_{T_2} < \cdots < w_k W_{T_k}$$

which in turn determines a sequence of parabolic subgroups

$$w_1 W_{T_1} w_1^{-1}, w_2 W_{T_2} w_2^{-1}, \dots , w_k W_{T_k} w_k^{-1}.$$

It is then natural to investigate the conjugacy classes of maximal parabolic subgroups. With these thoughts in mind, let k be a fixed positive integer. A Coxeter system (W, S) has *condition* k if every maximal simplex $A \in N^*(S)$ in the nerve has dimension $k - 1$. Equivalently, (W, S) has condition k if $rk(W_A) = k$ for every $A \in N^*(S)$.

Lemma 3.2.3 *Suppose that the Coxeter system (W, S) has condition k. If G is a maximal finite subgroup of W, then $rk(G) = k$.*

Proof: By Lemma 3.2.1, G is a parabolic subgroup of W, so $G = wW_T w^{-1}$ for some $w \in W$ and $T \in \mathcal{S}(S)$. If $T \in N^*(S)$ then we are done. Otherwise, T must be properly contained in some maximal simplex $A \in N^*(S)$. If this is the case, then G is properly contained in the finite parabolic subgroup $wW_A w^{-1}$, contradicting the maximality of G. □

Lemma 3.2.4 *Assume that the Coxeter system (W, S) has condition k. If $A \in N^*(S)$ is a maximal simplex in the nerve, then W_A is a maximal finite subgroup of W. In particular, the set*

$$F(W_A) = \{wW_T \in W\mathcal{S}(S) : W_A = wW_T w^{-1}\}$$

is equal to the conjugacy class of W_A in W.

Proof: Let G be a maximal finite subgroup of W such that $W_A \subseteq G$. By Lemma 3.2.1, G is a parabolic subgroup of W and, by Lemma 3.2.3, $rk(G) = rk(W_A) = k$. By [7, Lemma 1.4], $G = W_A$. □

As indicated, we intend to investigate the nature of the conjugacy classes of maximal parabolic subgroups in Coxeter systems which have condition k. To do so we need only consider the conjugacy classes corresponding to special subgroups of the form W_A, where $A \in N^*(S)$ is a maximal simplex in the nerve. With Lemma 3.2.4 in hand this is equivalent to investigating the fixed poset $F(W_A)$. The next result shows that, given an arbitrary Coxeter system (W, S) and a maximal simplex $A \in N^*(S)$ such that W_A is a maximal subgroup (eg., when (W, S) has condition k), the composition of $F(W_A)$ is quite simple.

Lemma 3.2.5 *Let (W, S) be an arbitrary Coxeter system and let $A \in N^*(S)$ be a maximal simplex in the nerve. In addition, suppose that W_A is a maximal finite subgroup of W. If $wW_T \in F(W_A)$, then $F(W_A)_{\leq wW_T}$ is a singleton.*

Proof: Applying Theorem 3.2.2 we see that the subposet $F(W_A)_{\leq wW_T}$ is isomorphic to the poset of faces of a convex cell of dimension $|T| - |A|$. Since

$wW_T \in F(W_A)_{\leq wW_T}$, the subposet $F(W_A)_{\leq wW_T}$ is nonempty. We conclude that $|T| - |A| \geq 0$ and so $|T| \geq |A|$. Since $W_A = wW_Tw^{-1}$, it follows that $w^{-1}W_A \in F(W_T)$ and, once again by Theorem 3.2.2, $|A| \geq |T|$, implying that $|A| = |T|$. Therefore the dimension of the aforementioned cell is zero and the result follows. □

The key result concerning the conjugacy classes of maximal special subgroups is the following.

Theorem 3.2.6 *Let (W, S) be an arbitrary Coxeter system and suppose that A and B are distinct simplices of $N(S)$. If W_A and W_B are maximal special subgroups of W, then W_A and W_B are not conjugate in W.*

Proof: Since W_A is a maximal finite subgroup of W, the fixed poset of W_A acting on $W\mathcal{S}(S)$ is given by

$$F(W_A) = \{wW_T : W_A = wW_Tw^{-1}\}.$$

It suffices to show that the complement $F(W_A) \setminus W_A$ is empty. To this end we suppose the contrary. Let $wW_T \in F(W_A)$ with $wW_T \neq W_A$. By Lemma 3.2.5, $F(W_A)_{wW_T\geq}$ is a singleton. It follows that Σ^{W_A} is a collection of 0-cells and in particular, contains no non-constant path. Let $\gamma : I \longrightarrow \Sigma$ be the geodesic arc connecting the vertices W_A and wW_T in Σ. Since W acts by isometry on Σ, the (unique) geodesic segment $[W_A, wW_T]$ is contained in the wall Σ^α for each $\alpha \in A$. By Lemma 2.3.6, $\bigcap_{\alpha\in A} \Sigma^\alpha = \Sigma^{W_A}$. This implies that

$$[W_A, wW_T] \subseteq \bigcap_{\alpha\in A} \Sigma^\alpha = \Sigma^{W_A},$$

which is impossible. □

Corollary 3.2.7 *Assume that the Coxeter system (W, S) has condition k. If (W, S') is another Coxeter system for W, then there exists a bijection*

$$\phi : N^*(S) \longrightarrow N^*(S')$$

between the maximal simplices of the associated nerves.

Proof: Let $A \in N^*(S)$ be a maximal simplex in $N(S)$. By Lemma 3.2.4, W_A is a maximal finite subgroup of W. Applying Lemma 3.2.1, we see that W_A is a S'-parabolic subgroup, so there is a $w \in W$ and an $A' \in \mathcal{S}(S')$ such that $W_A = wW_{A'}w^{-1}$. Theorem 3.2.6 implies that A' is unique. Since $W_{A'}$ is a maximal subgroup of W, it follows that A' is a maximal simplex in $N(S')$. By setting $\phi(A) = A'$, we have $W_A = wW_{\phi(A)}w^{-1}$, providing the desired bijection. □

3.3 The Schur Multiplier of a Coxeter Group

In [11], R. Howlett provides a remarkable algorithm for computing the second cohomology of a Coxeter group W given a system (W, S). This algorithm relies only on the isomorphism-type and labeling scheme for the Coxeter graph Γ_S.

Let G be a group and let $\mathbb{C}^\times$ be the multiplicative group of complex numbers with trivial G-action. The *Schur multiplier*, $Mult(G)$, of G is defined as

$$Mult(G) = H^2(G, \mathbb{C}^\times).$$

If (W, S, m) is a Coxeter system, let

$$A_2 = \{\{s, t\} \subseteq S : m(s, t) = 2\}.$$

write $\{s, t\} \cong \{s, t'\}$ if $\{s, t\}, \{s, t'\} \in A_2$ and $m(t, t')$ is odd. Let $\sim$ be the equivalence relation on A_2 generated by $\cong$. Let Γ'_S be the graph obtained by deleting from Γ_S the edges with even label. Define

$\mu_S =$ the number of edges of Γ_S with finite edge label;

$\nu_S =$ the number of equivalence classes of $\sim$ on A_2;

$\zeta_S =$ the number of connected components of Γ'_S.

Theorem 3.3.1 (Howlett [11]) *If* (W, S) *is any Coxeter system for* W, *then*

$$Mult(W) \cong \mathbb{Z}_2^{\mu_S + \nu_S + \zeta_S - |S|}.$$

S. Pride and R. Stöhr [16, Corollary, p. 62] prove a similar result in the case that (W, S) is "aspherical" (i.e., any three distinct elements generate an infinite subgroup of W).

3.4 Coxeter Graphs for Type K_n Groups

We now now intend to characterize the possible Coxeter graphs for Coxeter groups which have a Coxeter system of type K_n. Given such a system (W, S), let us take note of some of the properties it enjoys.

1. The nerve $N(S)$ is the complete graph on $n = |S|$ vertices. This implies that the Coxeter system has condition 2.

2. Each edge $\{s, t\}$ in the Coxeter graph Γ_S corresponds to a maximal simplex $\{s, t\}$ in the nerve $N(S)$.

3. For each edge $\{s, t\}$ in the Coxeter graph Γ_S, the corresponding special subgroup $W_{\{s,t\}}$ is a maximal finite subgroup of W and is isomorphic to a dihedral group of order $2m(s, t)$.

4. If $n = |S| \geq 3$, then for any $s \in S$, the special subgroup $W_{\{s\}}$ can be expressed as

$$W_{\{s\}} = \bigcap_{i=1}^{n-1} W_{A_i},$$

where $\{A_i\}_{i=1}^{n-1}$ is the collection of edges in Γ_S containing the vertex s. This is a direct application of Theorem 2.2.1 (b).

Given an arbitrary Coxeter system (W, S), recall that for each positive integer k, the set $P_k(S)$ is the collection of S-parabolic subgroups of W which have parabolic rank k. The following result is proved in [7].

Theorem 3.4.1 (R. Charney, M. Davis) *Suppose (W, S) and (W, S') are Coxeter systems for any Coxeter group W. If $P_1(S) = P_1(S')$, then $P_k(S) = P_k(S')$ for all $k \geq 1$.*

We use this to prove the following.

Theorem 3.4.2 *Let (W, S) be a Coxeter system of type K_n and let (W, S') be another Coxeter system for W. Then, for all $k \geq 1$, $P_k(S) = P_k(S')$.*

Proof: By Theorem 3.4.1 it suffices to show that $P_1(S) = P_1(S')$. We consider three cases:

Case 1. If $n = 1$, then the subgroup lattice of W consists of the trivial subgroup and W itself. Thus $P_1(S) = W = P_1(S')$.

Case 2. If $n = 2$, then W is isomorphic to the dihedral group D_{2k+1} for some $k \geq 1$ and so W is the full symmetry group of a regular $(2k+1)$-gon P in $\mathbb{R}^2$. Since $S = \{s_1, s_2\}$ generates W and each element of S has order 2, there are lines of symmetry L_1 and L_2 through P such that s_1 (resp. S_2) acts on P by reflection through the line L_1 (resp. L_2) (every symmetry of P with order 2 is a reflection because P has an odd number of sides), and L_1 and L_2 form an angle of $\dfrac{q\pi}{2k+1}$ where q and $2k+1$ are relatively prime. Note that each line of symmetry through P corresponds uniquely to a reflection $r \in R_S$. Since each $s' \in S'$ has order 2, s' acts on P by reflection through a line of symmetry $L_{s'}$ and is therefore conjugate in W to an element of S. Let $G \in P_1(S)$. Then $G = wW_{\{s\}}w^{-1}$ for some $s \in S$ and $w \in W$. Choosing any $s' \in S'$ we see that there is an element $v \in W$ such that $s = vs'v^{-1}$. It follows that

$$G = wW_{\{s\}}w^{-1} = wvW_{\{s'\}}(wv)^{-1},$$

implying that $G \in P_1(S')$ and so $P_1(S) \subseteq P_1(S')$. The same argument shows that $P_1(S') \subseteq P_1(S)$.

Case 3. Assume $n \geq 3$. Choose $G \in P_1(S)$. Then $G = wW_{\{s\}}w^{-1}$ for some $s \in S$ and $w \in W$. Letting $\{A_i\}_{i=1}^{n-1}$ be the collection of edges of Γ_S containing s we have

$$G = w\left(\bigcap_{i=1}^{n-1} W_{A_i}\right)w^{-1}$$

$$= \bigcap_{i=1}^{n-1}(wW_{A_i}w^{-1}).$$

By Corollary 3.2.7 we see that for each $1 \leq i \leq n-1$, the subgroup $wW_{A_i}w^{-1}$ is an S'-parabolic subgroup and, by Corollary 2.3.5, the intersection of parabolic subgroups is a parabolic subgroup. Therefore $G \in P_1(S')$, implying that $P_1(S) \subseteq P_1(S')$.

Now choose $G \in P_1(S')$. Then $G = vW_{\{t\}}v^{-1}$ for some $v \in W$ and $t \in S'$. Let $B \in N^*(S')$ be a maximal simplex in the nerve $N(S')$ such that $W_{\{t\}} \subseteq W_B$. By Corollary 3.2.7, $W_B = uW_Au^{-1}$ for some $u \in W$ and $A \in N^*(S)$ ($A = \phi^{-1}(B)$). The parabolic subgroup uW_Au^{-1} is isomorphic to a dihedral group D_{2k+1}, for some positive integer k. Also, $v^{-1}tv \in uWu^{-1}$ and has order 2, implying that $v^{-1}tv \in R_S$ (any element in D_{2k+1} of order 2 is a reflection). Hence $t \in R_S$. This implies that $W_{\{t\}} \in P_1(S)$ and therefore $G \in P_1(S)$. We now have $P_1(S') \subseteq P_1(S)$ which completes the proof in the case that $n \geq 3$.

Thus for every positive integer n we have the desired result. □

As a consequence we have:

Corollary 3.4.3 *Let (W, S) be a Coxeter system of type K_n and let (W, S) be any other Coxeter system for W. Let $\phi : N^*(S) \longrightarrow N^*(S')$ be the bijection as in Corollary 3.2.7. Then $\{a, b\}$ is an edge in $N(S)$ if and only if $\phi(\{a, b\})$ is an edge in $N(S')$.*

Proof: The edge $\{a, b\}$ corresponds to the S-parabolic subgroup $W_{\{a,b\}}$ which has parabolic rank 2. By Corollary 3.2.7, $W_{\{a,b\}} = wW_{\phi(\{a,b\})}w^{-1}$ for some $w \in W$ and, by Theorem 3.4.2, $wW_{\phi(\{a,b\})}w^{-1}$ has S'-parabolic rank 2 as well. In other words, $\phi(\{a, b\})$ is an edge in $N(S')$. The proof of the reverse implication is similar. □

Though Corollary 3.2.7 determined a bijection between the maximal simplices of the respective nerves, prior to Theorem 3.4.2 we were unable to decide whether the bijection was dimension preserving.

Given a Coxeter system (W, S) of type K_n and any other Coxeter system for W, we now intend to show that the Coxeter graphs Γ_S and $\Gamma_{S'}$ are weakly isomorphic. This is accomplished by establishing that the number of vertices and edges of $\Gamma_{S'}$ are the same as those of Γ_S.

Lemma 3.4.4 *Let (W, S, m) be a Coxeter system of type K_n. If (W, S', m') is any other Coxeter system for W and $\{p, q\} \in \mathcal{S}(S')$, then $m'(p, q)$ is odd.*

Proof: Suppose $m'(p, q)$ is even for some $\{p, q\} \in \mathcal{S}(S')$. In this case $W_{\{p,q\}}$ cannot be a maximal finite subgroup of W (every maximal finite subgroup of W is isomorphic to a dihedral group D_{2k+1} for some integer $k \geq 1$). By some Lemma, if $B \in \mathcal{S}(S')$ is such that $W_{\{p,q\}}$ is properly contained in W_B, then $|B| \geq 3$. But by Theorem 3.4.2, $P_k(S') = P_k(S) = \emptyset$ for $k \geq 3$. Thus $m'(p, q)$ must be odd, contradicting our assumption that $m'(p, q)$ is even. □

Recall that if (W, S, m) is a Coxeter system for W, then μ_S is the number of edges of the Coxeter graph with finite edge label.

Lemma 3.4.5 *Let (W, S, m) be a Coxeter system of type K_n. If (W, S', m') is another Coxeter system for W, then $\mu_S = \mu_{S'}$.*

Proof: Let $\{a, b\}$ be an edge in Γ_S. Since $m(a, b)$ is finite, $\{a, b\}$ is an element of $N^*(S)$. By Corollaries 3.2.7 and 3.4.3, the edge $\{a, b\}$ corresponds uniquely to an edge $\phi(\{a, b\}) \in N^*(S')$, so $\phi(\{a, b\})$ is an edge in $\Gamma_{S'}$ with finite edge label. Therefore $\mu_{S'} \geq \mu_S$. If $\{p, q\}$ is an edge in $\Gamma_{S'}$ with $m'(p, q) \neq \infty$, the previous argument reverses, showing that $\mu_{S'} \leq \mu_S$. □

Lemma 3.4.6 *Suppose (W, S, m) is a Coxeter system of type K_n. If (W, S', m') is another Coxeter system for W, then $|S| = |S'|$.*

Proof: By Lemma 3.4.4, $m'(s, t)$ is odd for all $\{s, t\} \in \mathcal{S}(S')$. From this we conclude that $\nu_{S'} = 0$ and $\zeta_{S'} = 1$ (there are no even edge labels on Γ_S). By Theorem 3.3.1,

$$\mathbb{Z}_2^{\mu_S + 1 - |S|} \cong Mult(W) \cong \mathbb{Z}_2^{\mu_{S'} + 1 - |S'|}.$$

Lemma 3.4.5 forces $|S| = |S'|$. □

Theorem 3.4.7 *If (W, S) be a Coxeter system of type K_n and (W, S') is any other system for W, then (W, S') is of type K_n. Moreover, the multi-sets of edge labels of Γ_S and $\Gamma_{S'}$ are equal.*

Proof: Since $\Gamma_{S'}$ does not contain a loop (an edge with a single vertex), Lemmas 3.4.6 and 3.4.5 imply that $\Gamma_{S'}$ is complete on $n = |S|$ vertices. Lemma 3.4.4 implies that each edge label is odd. In other words, (W, S') is of type K_n. Corollaries 3.2.7 and 3.4.3 provide a one-to-one correspondence $\phi : N^*(S) \longrightarrow N^*(S')$ between the edges of $N(S)$ and $N(S')$. Each edge $\{a, b\}$ in $N(S)$ determines a dihedral group $W_{\{a,b\}}$ of order $2m(a, b)$ which corresponds uniquely to the dihedral group $W_{\phi(\{a,b\})}$ which is also of order $2m(a, b)$, which proves the second assertion of the theorem. □

Corollary 3.4.8 *For any $k \geq 1$, the dihedral group D_{2k+1} is rigid.*

Corollary 3.4.8 is merely a particular case of the following result.

Corollary 3.4.9 *Let (W, S) be a Coxeter system of type K_n such that at most one edge label differs from the others. Then W is rigid.*

Proof: Let (W, S) be another Coxeter system. By Theorem 3.4.7, the graphs Γ_S and $\Gamma_{S'}$ are weakly isomorphic and the multi-sets of edge labels agree. Since all but at most one edge label is identical, Γ_S and $\Gamma_{S'}$ are clearly isomorphic. $\square$

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