Solution of a PDE Boundary Value Problem

Calculate an approximation to the following problem:
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in the Moon Region shown on the right.}
\]
The boundary conditions are \( u(x,y) = 0 \) on the outside unit radius circle \( x^2 + y^2 = 1 \) and \( u(x,y) = 1 \) on the inside circle \( \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4} \).

This problem can be solved analytically using conformal mapping methods (not covered in this course). The exact solution is
\[
u(x,y) = \frac{1 - x^2 - y^2}{(x - 1)^2 + y^2}.
\]

This is a challenging numerical problem since the solution is unbounded as the point \((1,0)\) is approached. One would expect Finite Difference and Galerkin’s methods to require some care.

Finite Differences are probably best implemented using polar coordinates for the PDE:
\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \Leftrightarrow \quad \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.
\]
The region is symmetric and would then be discretized along \( r \) and \( \theta \), where the top region is the union:
\[
\left\{(r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2}, \cos \theta \leq r \leq 1 \right\} \cup \left\{(r, \theta) \mid \frac{\pi}{2} \leq \theta \leq \pi, 0 \leq r \leq 1 \right\}
\]
Don’t use the origin nor the \((1,0)\) points with finite differences. Galerkin’s methods may be easier for this problem especially selecting suitable expansion functions. Harmonic expansion functions are best since then Laplace’s PDE will be satisfied and the coefficients are then selected to best satisfy the boundary conditions. Use the sum of two line integrals for the boundaries and they will require parametric equations. Recall that parametric equations for the large circle are: \( x = \cos t, \ y = \sin t, \ 0 \leq t \leq 2\pi \), and for the inside circle are
\[
x = \frac{1}{2}(1 + \cos t), \ y = \frac{1}{2}\sin t, \ 0 \leq t \leq 2\pi
\]
A line integral over a curve \( C \) is given by:
\[
\int_C f(x,y) \, ds = \int_{t_1}^{t_2} f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt,
\]
where \( x = x(t), \ y = y(t), \ t_1 \leq t \leq t_2 \) are the parametric equations of the curve.

In your report include integral error measures (approximate double integral for Finite Differences):

\[
\text{Galerkin’s: Error} = \left( \int_{\text{Outside Circle}} \| \hat{u}(x,y) \|^2 \, ds + \int_{\text{Inside Circle}} \| \hat{u}(x,y) - u(x,y) \|^2 \, ds \right)^{\frac{1}{2}}
\]
\[
\text{Finite Differences: Error} = \left( \int_{\text{Moon Region}} \| u(x,y) - \hat{u}(x,y) \|^2 \, dA \right)^{\frac{1}{2}}
\]

Also include a listing of your implementations and your analysis and conclusions. You can select to implement one of Finite Differences with different grid sizes or Galerkin’s with different expansion functions. However, write an analysis contrasting the two methods and why you selected the one implemented.