# Vector Spaces

## Vector Space Definition:
A set of vectors \( V = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \) such that addition and scalar multiplication are closed. This means:

i) Vectors equal to a scalar times any vector in the set, are also in the set. Mathematically:
\[
\text{if } \vec{v} \in V \text{ and } c \in \mathbb{R} \Rightarrow c\vec{v} \in V
\]

ii) Vectors that are sums of any vector in the set, are also in the set. Mathematically:
\[
\text{if } \vec{v}, \vec{w} \in V \Rightarrow \vec{v} + \vec{w} \in V
\]

## Comments:
- The set of vectors can be ordinary \( n \) component vectors, functions, polynomials, matrices, etc. Example: \( V = \{ \text{all polynomials up to degree 2} \} \)
- The set of vectors may include equations defining their characteristics instead of a list. Examples:
\[
V = \left\{ y(t) \left| \begin{array}{l}
x + y + z = 0 \\
y = 0
\end{array} \right. \right\}
\]
\[
V = \mathbb{R}^4
\]
- The scalar multiplication requirement must be met with the constant \( c \) negative, zero and positive.
- The addition requirement must be met with \( \vec{v} + \vec{w} = \vec{v} + \vec{w} \) also (addition is commutative).

## Example problems:
- Show \( V = \{ \text{all polynomials up to degree 2} \} \) is a vector space
  Two polynomials in this vector space are \( p_1 = a_0 + a_1t + a_2t^2 \) and \( p_2 = b_0 + b_1t + b_2t^2 \) with the coefficients being any real constants. To prove this is a vector space:
  i) \( cp_1 = ca_0 + ca_1t + ca_2t^2 \in V \) since \( c \) times any coefficient is a real constant coefficient.
  ii) \( p_1 + p_2 = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 \in V \) and the coefficients remain real constant coefficients.

- Show \( V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right| x + y + z = 0 \right\} \) is a vector space.
  Two vectors in this space are \( \vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \), and \( x_1 + y_1 + z_1 = 0 \) and \( \vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \), and \( x_2 + y_2 + z_2 = 0 \)
  To prove this is a vector space we need to verify the properties:
  i) \( c\vec{v} = cx_1 + cy_1 + cz_1 = c(x_1 + y_1 + z_1) = 0 \in V \)
  \( \vec{v} + \vec{w} = (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \)
  ii) \( (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0 \in V \)
- Is \( V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \right| x^2 + y^2 = 1 \} \) a vector space?
  This space fails both requirements, but scalar multiplication is easiest: \( (cx)^2 + (cy)^2 = c^2(x^2 + y^2) = c^2 \neq 1 \), for all \( c \in \mathbb{R} \)
Span of a set of vectors

**Span Definition:**
Written Span \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} of vector space \(V\) is the set of vectors described (spanned, covered) by a linear combination of the set of vectors. This means:

i) Any vector \(\vec{v} \in V\) described by \(\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n\) is in the set, and thus this equation ‘describes the span’

**Comments:**
- Note that Span \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} does not mean that all the vectors in \(V\) are spanned. Often an exercise asks if the Span is all of \(V\).
- To show a span is equal to all of \(V\) an REF of the linear combination of the vectors is often the easiest way to do it. An REF of the system of equations provides the answer with a general vector in \(V\). When the system of equations yields a square matrix, a non-zero determinant means a unique solution exists. A unique solution is not necessary, however: vectors that result in rectangular systems of equations with many solutions is OK, but any systems were REF produces a zero row on the left and some combinations of the components of the typical vector in \(V\) on the right imply the set does not span all of \(V\) because the only way to have a solution in this case is with the right hand side being 0 and this implies a subset of the space.
- Recall that the set of vectors can be ordinary \(n\) component vectors, functions, polynomials, matrices, etc.

**Example problems:**
- Show Span \(\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mathbb{R}^2\)

  Begin with a general vector \(\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2\) and span means

  \[
  \begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},
  \]

  now if can we solve for \(c_1\) and \(c_2\) we have proven the set spans all of \(\mathbb{R}^2\). An REF of the system of equations provides the answer. For square systems like this one, a non-zero determinant means a unique solution exists; and since this determinant is non-zero, this set of vectors span all of \(\mathbb{R}^2\).

- Show Span \(\{1, t, 2t^2 - 1\} = \Pi_2\) (all polynomials up to degree 2).

  We must show that any polynomial up to degree 2: \(p = a_0 + a_1t + a_2t^2\) must be expressed as a linear combination:

  \[
  a_0 + a_1t + a_2t^2 = c_1 + c_2t + c_3(2t^2 - 1) = (c_1 - c_3) + c_2t + 2c_3t^2.
  \]

  Equating coefficients: \(a_0 = (c_1 - c_3),\ a_1 = c_2,\ a_2 = 2c_3\), which is a linear system of equations:

  \[
  \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}\]

  that has a solution since the determinant is 2.
### Linear Dependence/Independence of a set of vectors

**Linear Dependence:**
A set of vectors \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \) is dependent if any one of the vectors can be written as a linear combination of the others. A mathematical statement for this is: if the equation
\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n = \vec{0}
\]
has any solutions for the coefficients:
\( c_1, c_2, \ldots, c_n \), where not all are zero, then the vectors are dependent. If on the other hand, the only solution is
\( c_1 = c_2 = \ldots = c_n = 0 \), then the vectors are independent.

When the vectors are functions, of \( t \) for example, the definition includes an interval. That is, the set of vectors (functions) are dependent if
\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n = \vec{0}
\]
has any solutions for the coefficients:
\( c_1, c_2, \ldots, c_n \), where not all are zero, over the entire interval of \( t \) of interest.

**Comments:**
- For ordinary vectors, the equation
  \( c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n = \vec{0} \) is a homogeneous system of equations.
- If this system is square (\( n \times n \)), the determinant of the matrix with columns being the vectors can be used to determine dependence or independence. If this determinant is zero, vectors are dependent (the homogeneous system of equations has many non-zero solutions); if the determinant is not zero, vectors are independent (non-zero determinant means we can multiply the equation by the inverse resulting in the solution being all zero).
- If this system is not square, use REF to determine dependence or independence. If REF results in free columns, the vectors are dependent. If REF results in no free columns, then the vectors are independent.
- For functions, of \( t \) for example, the equation
  \( c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n = \vec{0} \) may not be a system of equations. One way of creating a system of equations is to select \( n \) distinct values of \( t \) to generate \( n \) equations. This proves independence if the only solution is
  \( c_1 = c_2 = \ldots = c_n = 0 \)
(dependence is not proven, however, if non-zero solutions exist). A second way is possible if we can differentiate the equation \( n \) times creating \( n \) equations. The determinant of the matrix is called the Wronskian. If the Wronskian is not zero, the functions are independent, and when it is zero the functions are dependent.

**Example problems:**
- Show the set \( V = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \) is independent
  The equation is
  \[
  c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
  \]
  Since the determinant \( = 2 \), non-zero, we can multiply the matrix equation by the inverse and readily see that the unique (only) solution is \( c_1 = c_2 = 0 \). Therefore the vectors are independent.
- Show the set \( V = \{ 1, t, 2t^2 - 1 \} \) is independent
  The equation is
  \[
  c_1 + c_2 t + c_3 (2t^2 - 1) = (c_1 - c_3) + c_2 t + 2c_3 t^2 = 0.
  \]
  Since this polynomial is 0 for all \( t \), the coefficients must be zero:
  \( c_1 - c_3 = 0, \ c_2 = 0, \ 2c_3 = 0 \), which is the system of equations
  \[
  \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \]
  Again, the determinant is 2, and the only solution is \( c_1 = c_2 = c_3 = 0 \). We can also use the Wronskian:
  \[
  \begin{bmatrix} 1 & t & 2t^2 - 1 \\ 0 & 1 & 4t \\ 0 & 0 & 4 \end{bmatrix} = 4, \]  which also shows the set is independent. The harder approach is to select three values of \( t \), say \( t = 0, t = 1, \) and \( t = -1 \) to generate the three equations:
  \[
  c_1 - c_3 = 0, \ c_1 + c_2 + c_3 = 0, \ c_1 - c_2 + c_3 = 0, \]  which yield the unique solution \( c_1 = c_2 = c_3 = 0 \) with the same answer.
Basis Definition:
A set of vectors \( V = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \) is a basis, or forms a basis, of a vector space, when

i) the vectors are independent, and

ii) the span of the vectors equals the vector space.

A vector space may have many sets of vectors that can be a basis. The Dimension of a vector space \( V \) is the number of vectors in a basis. This number is the same for all the basis of a vector space. Some vector spaces, like the space of all polynomials, have infinite dimension.

Comments:
- Typical exercises ask to show whether a set of vectors is a basis. We answer these questions by showing the vectors are independent and they span the vector space (see other summaries).
- If you suspect the set of vectors is not a basis, you should see if one vector is obviously equal to a combination of one or two other vectors, which would mean the set is dependent and could not be a basis.
- We are most familiar with the standard basis for ordinary vectors. For example the standard basis for \( \mathbb{R}^3 \) is the set
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Example problems:
- Show \( V = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} \) is a basis for \( \mathbb{R}^2 \)
  
  We already showed before that this set is independent, and that the set spans \( \mathbb{R}^2 \). Therefore, this set is a basis for \( \mathbb{R}^2 \).

- Determine if the set \( V = \{1, t, t+1\} \) is a basis for all polynomials up to degree 2.
  
  We suspect not, and verify by inspection that the third polynomial is the sum of the first two, thus they are dependent, and the set cannot be a basis.

- Show \( V = \{\cos t, \sin t\} \) is a basis for the solutions of \( y'' + y = 0 \)
  
  First we show independence:
  
  Wronskian = \[
  \begin{vmatrix}
  \cos t & \sin t \\
  -\sin t & \cos t
  \end{vmatrix} = \cos^2 t + \sin^2 t = 1, \text{ not zero for all } t,
  \]
  
  so the set is independent. Now to show the set is a basis we show that the Span: \( y(t) = c_1 \cos t + c_2 \sin t \) represents the solutions to the differential equation. This is true which can be verified using the characteristic polynomial method to find the solution. The characteristic polynomial is: \( r^2 + 1 = 0, \ r = \pm i \) with solutions being the span of the given vectors. Thus the set is a basis.
## Linear Transformations

**Definition:**
Linear Transformations, also called mappings, are similar to functions and vector functions we studied in algebra and Calculus courses. A Linear Transformation operates, transforms or maps, a vector from one vector space into a vector in possibly a different vector space. The ‘Linear’ implies some rules must apply.

A Linear Transformation from vector space \( V \) into vector space \( W \) is written: \( T : V \rightarrow W \). If we define a vector in \( V \) by \( \vec{v} \) and a vector in \( W \) by \( \vec{w} \), we write \( T(\vec{v}) = \vec{w} \) (notice the similarity with functions).

To be a Linear Transformation, the following two requirements must be met:

1. \( T(cv) = cT(v) \), \( v \in V \) and \( c \in \mathbb{R} \)
2. \( T(\vec{v} + \vec{u}) = T(\vec{v}) + T(\vec{u}) \), \( \vec{v}, \vec{u} \in V \)

Important properties of Linear Transformations to be covered later are:
- Image, written \( Im(T) \), and its dimension
- Kernel, written \( Ker(T) \), and its dimension
- Dimension of the vector space being mapped, written \( \text{dim}(V) \)

**Comments:**
- Recall that the set of vectors can be ordinary \( n \) component vectors, functions, polynomials, matrices, etc.
- The scalar multiplication requirement must be met with the constant \( c \) negative, zero and positive.
- The addition requirement must be met with \( \vec{u} + \vec{v} = \vec{v} + \vec{u} \) and \( T(\vec{u}) + T(\vec{v}) = T(\vec{v}) + T(\vec{u}) \) also (addition is commutative).
- Linear Transformation matrices are very important to determine. These matrices allow the calculation and determination of the Image of the transformation: \( Im(T) \), the Kernel of the transformation: \( Ker(T) \), and the dimension of the vector space being mapped.
- Linear Transformation of ordinary vectors have a geometric interpretation and application. Transformation matrices can be interpreted as rotations, shears, etc. of objects about the origin.
- Common Linear Transformations (also called operators) include: Derivative:
  \[ D^n(f) = \frac{d^n}{dx^n} f \]
  Integral:
  \[ I(f) = \int_a^b f(t)dt \]
  Differential:
  \[ L_n(y) = y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y \]
  Transforms:
  \[ F(f) = \int_{-\infty}^{\infty} e^{i\omega t} f(t)dt \]

**Example problems:**
- Determine if \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \), \( T(x,y) = (x,1,y,1) \) is a linear transformation.
  We can also write these transformations using vector notation as follows:
  \[
  T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 1 \\ y \\ 1 \end{bmatrix}
  \]
  To be linear we must meet the two requirements. Checking:
  \[
  T(cx, cy) = (cx, 1, cy, 1) = c(x, 1, y, 1) = cT(x, y)
  \]
  \[
  T(\vec{u} + \vec{v}) = T(\vec{v} + \vec{u}) = T(\vec{v}) + T(\vec{u})
  \]
  Thus this is a Linear Transformation.
- Determine the transformation matrix \( A \) such that the linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \), \( T(x,y) = (x+2y, x-2y, y) \) can be written as \( T(\vec{u}) = A\vec{u}, \ \vec{u} \in \mathbb{R}^2 \)
  For this problem it is easier to work with vector notation:
  \[
  T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ x - 2y \\ y \end{bmatrix}
  \]
  and thus the transformation matrix is
  \[
  A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}
  \]
Consider the Linear Transformation to be: 
\[ T : V \rightarrow W, \quad \vec{v} \in V \text{ and } \vec{w} \in W \text{ and } \]
whenever possible: \( T(\vec{v}) = A\vec{v} = \vec{w} \) where \( A \) is the transformation matrix.

**Image (or Range):** \( \text{Im}(T) \)

\( \text{Im}(T) \) is the set of all the vectors in \( W \) mapped by the transformation: \( \vec{w} = T(\vec{v}) \). With the transformation matrix \( A \), the \( \text{Im}(T) \) is the Span of the set of vectors from the columns of \( A \): 
\[ \vec{w} = A\vec{v} \]. The dimension of the image, \( \dim(\text{Im}(T)) \), is called rank(\( T \)), and is equal to the number of pivot columns in the RREF of \( A \). \( T \) is surjective (onto) when the \( \text{Im}(T) \) is equal to \( W \).

**Kernel (or nullspace):** \( \text{Ker}(T) \)

\( \text{Ker}(T) \) is the set of all vectors in \( V \) that are mapped to zero in \( W \): \( T(\vec{v}) = \vec{0} \). With the transformation matrix \( A \), the \( \text{Ker}(T) \) is the set of vectors that is equal to all the solutions of \( A\vec{v} = \vec{0} \). The zero vector is always in the set, since \( \vec{v} = \vec{0} \) is a solution. The dimension of the kernel, \( \dim(\text{Ker}(T)) \), is called nullity(\( T \)), and is equal to the number of free columns in the RREF of \( A \) (note that the zero vector is not counted). \( T \) is injective (1-1) when the \( \text{Ker}(T) \) is just the zero vector, or nullity(\( T \)) = 0.

**Dimension:**
We saw earlier that the Dimension of a vector space is equal to the number of vectors in a basis for the vector space. The dimension is also equal 
\[ \dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T)) \]
to:
\[ \text{nullity}(T) + \text{rank}(T) \]

**Comments:**
- Nonhomogeneous Principle for Linear Transformations is similar to the superposition principle for linear equations. Namely, given a non-homogeneous problem: \( T(\vec{x}) = \vec{b} \) and \( \vec{x} \) being any particular solution, the general set of solutions is \( \vec{x} = \vec{x}_h + \vec{x}_p \), where \( \vec{x}_h \) is \( \text{Ker}(T) \).
- The Transformation matrix answers all property questions with the Image equation.
  i) When the matrix is square, a non-zero determinant means a unique solution exists. This means the \( \text{Ker}(T) \) is null with only the zero vector, thus nullity is zero. In addition \( \text{Im}(T) \) is equal to \( W \), which means rank(\( T \)) is the number of rows in the matrix. The transformation is surjective and injective. If determinant is zero, calculate REF and check below.
  ii) If REF yields free columns, then the \( \text{Ker}(T) \) is not null, and nullity is equal to the number of free columns (not injective). If there are no free columns, then \( \text{Ker}(T) \) is null with only the zero vector, thus nullity is zero (injective).
  iii) If REF yields no zero rows, \( \text{Im}(T) \) is equal to \( W \) and the transformation is surjective. The presence of zero rows typically means \( \text{Im}(T) \) is equal to some subspace. It could still be equal to \( W \) if the \( \text{Im}(T) \) subspace matches the definition of \( W \). The number of pivot columns is rank(\( T \)) = dim(\( \text{Im}(T) \)).

**Example problem:**
- Calculate the Image, Kernel and all dimensions of the Linear Transformation 
  \[ T : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad T(\vec{v}) = A\vec{v} = \vec{w}, \text{ with the matrix } A \]
given by \( A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix} \).

\( \text{Im}(T) \) is given by 
\[ \begin{align*}
  w_1 &= 1 \quad 1 \quad 1 \quad v_1 \quad \begin{bmatrix} v_1 + v_2 + v_3 \end{bmatrix} \\
  w_2 &= 1 \quad 2 \quad 1 \quad v_2 \quad \begin{bmatrix} v_1 + 2v_2 + v_3 \end{bmatrix} \\
  w_3 &= 2 \quad 3 \quad 2 \quad v_3 \quad \begin{bmatrix} 2v_1 + 3v_2 + 2v_3 \end{bmatrix}
\end{align*} \]
for any real constants \( v_1, v_2, \) and \( v_3 \).

\( \text{Ker}(T) \) are all the solutions of 
\[ \begin{align*}
  &1 & 1 & 1 & v_1 &= 0 \\
  &1 & 2 & 1 & v_2 &= 0 \\
  &2 & 3 & 2 & v_3 &= 0
\end{align*} \]
with RREF: 
\[ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
and this means solutions are:
\[ \begin{bmatrix} v_1 \\ -v_3 \\ -1 \end{bmatrix} = r \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad r \in \mathbb{R} \]

Thus \( \text{Ker}(T) = \begin{bmatrix} r \\ 0 \\ 1 \end{bmatrix}, \quad r \in \mathbb{R} \). We see that nullity(\( T \)) = 1, rank(\( T \)) = 2 (number of pivot columns), and \( \dim(V) = 3 \). This transformation is neither injective nor surjective.